<u>Problem 1</u> (60 minutes)
Consider consumers i, denoted by superscripts, with consumption set $X^i = \mathbb{R}^2_+$. $(p, m) = (p_1, p_2, m)$ denotes price-income combinations with $p_1 > 0, p_2 > 0, m > 0$.

(A) A consumer with utility function $U^A(x_1, x_2) = x_1 x_2$ has absolute marginal rate of substitution

$$|MRS^A| = \frac{x_2}{x_1} \tag{1}$$

at strictly positive (internal) consumption bundles $(x_1, x_2) \gg 0$ — which you need not show.

Your Task: Use (1) and the tangency condition to show for the consumer's Marshallian demand $x^A(p, m) = (x_1^A(p, m), x_2^A(p, m))$:

(i) $x_1^A(p, m) < x_2^A(p, m)$ if $p_1 > p_2$.

(ii) $x_1^A(p, m) > x_2^A(p, m)$ if $p_1 < p_2$.

(iii) $x_1^A(p, m) = x_2^A(p, m)$ if $p_1 = p_2$.

(B) A consumer with utility function $U^B(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ has absolute marginal rate of substitution

$$|MRS^B| = \sqrt{\frac{x_2}{x_1}} \tag{2}$$

at strictly positive (internal) consumption bundles $(x_1, x_2) \gg 0$ — which you need not show.

Your Task: Use (2) and the tangency condition to show for the consumer's Marshallian demand $x^B(p, m) = (x_1^B(p, m), x_2^B(p, m))$:

(i) $x_1^B(p, m) < x_2^B(p, m)$ if $p_1 > p_2$.

(ii) $x_1^B(p, m) > x_2^B(p, m)$ if $p_1 < p_2$.

(iii) $x_1^B(p, m) = x_2^B(p, m)$ if $p_1 = p_2$.

(C) Now consider a pure exchange economy consisting of $n^A \geq 1$ consumers with utility function U^A and $n^B \geq 1$ consumers with utility function U^B. The aggregate or social endowment has the form $\omega = (k, k)$ with $k > 0$.

Question: For which prices p_1 is $(p_1, 1)$ a market clearing price system? Explain your answer!
You may apply the findings in (A) and (B) to all consumers to find and explain the answer.

(D) Like in (C), suppose that the aggregate endowment is of the form $\omega = (k, k)$ with $k > 0$.

Question: What is the equilibrium consumption of consumer i who has endowment bundle $\omega^i = (\omega_1^i, \omega_2^i)$?

Problem 2 (50 minutes)
A finite number k of individuals $i = 1, \ldots, k$ make a joint effort to obtain a profit $x \geq 0$ which is equally shared by the members of the group. The effort level of individual i is measured by a number $s_i \in [0,1] \equiv S_i$. The profit is determined by a yield function,

$\pi : S_1 \times \ldots \times S_k \to \mathbb{R}_+$,

i.e., $x = \pi(s_1, \ldots, s_k)$ is the profit resulting from the joint effort $s = (s_1, \ldots, s_k)$.

Each individual i has a utility function $v_i : S_i \times \mathbb{R}_+$ which represents i's preferences over effort-income pairs (s_i, x_i).

The joint decision problem is adequately described by the game

$$\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$$

where

$I = \{1, 2, \ldots, k\}$,
$S_i = [0,1]$ for $i \in I$,
$u_i(s_1, \ldots, s_k) = v_i(s_i, \frac{1}{k}\pi(s_1, \ldots, s_k))$ for all i, $s = (s_1, \ldots, s_k) \in S$.

DETERMINE the Nash equilibria (in pure strategies) for the special case

$\pi(s_1, \ldots, s_k) = \ln\left(1 + \sum_{i=1}^{k} s_i\right)$,
$v_i(s_i, x_i) = x_i - \frac{1}{2}s_i^2$,
$u_i(s_1, \ldots, s_k) = \frac{1}{k}\ln\left(1 + \sum_{j=1}^{k} s_j\right) - \frac{1}{2}s_i^2$.

HINT. A quadratic equation in $\sum = \sum_{j=1}^{k} s_j$ should be obtained and solved.

Problem 3 (70 minutes)
Consider the case of three goods labeled $i = 1, 2, 3$ and a profit maximizing and price-taking firm that produces good 3 whose price is $p = 1$. The firm has to choose one of two technologies:

> *Technology 1* produces good 3 with good 1 as input or factor of production. It is given by the production function
>
> $$f_1(z_1) = 4z_1^{1/2} \text{ for } z_1 \geq 0.$$
>
> The price of input 1 is $w_1 > 0$.
>
> *Technology 2* produces good 3 with good 2 as input or factor of production. It is given by the production function
>
> $$f_2(z_2) = 3z_2^{2/3} \text{ for } z_2 \geq 0.$$
>
> The price of input 2 is $w_2 > 0$.

(a) Determine the firm's maximum profit when it uses Technology 1.

(b) Determine the firm's maximum profit when it uses Technology 2.

(c) Determine which technology the profit maximizing firm is going to choose.

(d) In particular, which technology does the firm choose if $w_1 = w_2$?

(e) How would a change from $p = 1$ to $p \neq 1$ affect your answer?
No need to redo all the calculations!

<u>Problem 4</u> (60 minutes)
Pamela has the utility-of-wealth function $U(w) = \ln(1+w)$ (called Bernoulli utility function in the textbook and von Neumann-Morgenstern utility function by others) and a current wealth of 4. She can invest a fraction $\alpha \in [0,1]$ of her initial wealth in a risky project. This investment has revenue

- zero with probability $\theta \in (0,1)$;
- 12α with probability $1-\theta$.

In addition, Pamela keeps the amount $4(1-\alpha)$, the part of her initial wealth she does not invest.

(a) Is Pamela risk-averse? Explain!

(b) Let $E(\alpha) = \theta U(4(1-\alpha)) + (1-\theta)U(4(1-\alpha)+12\alpha)$,
Pamela's expected utility as a function of the chosen α.
Show that $E'(\alpha)$ is decreasing in α.

(c) For which values of θ is Pamela willing to assume some risk, that is, her optimal choice is an $\alpha^* > 0$. (Check $E'(0)$. Why?)

(d) For which values of θ does Pamela invest her entire initial wealth? (Check $E'(1)$. Why?)

(e) Suppose we know that Pamela is an expected utility maximizer with utility-of-wealth function $U(w) = \ln(1+w)$ and she can make an investment as described above. A priori, we do not know θ, but can infer θ from Pamela's optimal choice of $\alpha \in (0,1)$. What is the value of θ if Pamela's optimal choice is $\alpha^* = 1/2$?

Question 1 (20 minutes)

(A) Consider the simple Normal model (one parameter):

$$X_t \backsim \text{NIID}(\mu, \sigma^2),\ t{=}1, 2, ..., n, ..., \tag{1}$$

with σ^2 is known; 'NIID' stands for 'Normal, Independent and Identically Distributed'.

The sampling distribution of $\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$, where $\overline{X}_n{=}\frac{1}{n}\sum_{t=1}^{n} X_t$, is often stated in textbooks as:

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \backsim \mathsf{N}\left(0, \frac{\sigma^2}{n}\right) \tag{2}$$

(a) Explain why this is misleading by explaining the mean of the sampling distribution in (2), and relate your answer to the reasoning underlying estimation.

(b) Using your answer in (a) construct a $(1{-}\alpha)$ two-sided Confidence Interval (CI) for μ.

(c) Explain why the $(1{-}\alpha)$ probability cannot be assigned to the observed CI.

(B) Consider the null and under the alternative hypotheses:

$$H_0\colon\ \mu \leq \mu_0, \text{ vs. } \quad H_1\colon\ \mu > \mu_0, \tag{3}$$

in the context of the simple Normal model in (1).

(a) Compare and contrast the reasoning underlying hypothesis testing and use it explicitly to derive the sampling distributions of the test statistic $d(\mathbf{X}){=}\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\sigma}$, under both the null and the alternative hypotheses in (3).

(b) Define the optimal Neyman-Pearson (N-P) test based on $d(\mathbf{X})$ and explain what 'optimal' means in terms of its relevant properties.

(c) Compare and contrast the concepts of (i) type I error probability and (ii) the p-value.

(C) (a) State and explain the Strong Law of Large Numbers (SLLN) as it relates to $\overline{X}_n{=}\frac{1}{n}\sum_{t=1}^{n} X_t$ in (A).

(b) Using your answer in (a) explain why the SLLN does not justify the claim:

$$\overline{x}_n{=}\tfrac{1}{n}\textstyle\sum_{t=1}^{n} x_t \simeq \mu^*\text{-the true value of } \mu, \text{ for a large enough } n.$$

(c) Explain the difference between finite sample properties for estimators, such as unbiasedness and full efficiency and asymptotic properties such as consistency (strong and weak) and asymptotic Normality.

Question 2

(suggested time: 20 minutes)

Consider the following CLRM for a sample of n observations:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where the error term has the usual CLRM properties.

Assume the dependent variable measures per-household, per-week gasoline consumption (in gallons) for private vehicles.

The matrix $\mathbf{X}$ of explanatory variables contains the following:

1. C: vector of ones (intercept term, as usual)
2. CARS: number of private, gasoline powered vehicles owned by household
3. DRIVERS: number of household members of driving age
4. RURAL: binary indicator for rural households (1=rural, 0=urban)
5. PRICE: average weekly regional gasoline price (dollars per gallon)

The corresponding coefficients are labeled β_1 through β_5.

Part (a), 8 points

For the following linear hypotheses, to be tested via an F-test, show the form of $\mathbf{R}$ and $\mathbf{q}$ when the hypothesis is written in the form of $H_0: \ \mathbf{R}\boldsymbol{\beta} = \mathbf{q}$, where $\boldsymbol{\beta}$ is the full vector of coefficients: (note: $\mathbf{q}$ can be a vector or a scalar, so the bold-facing is just a general representation).

a An additional car has the same effect on gasoline consumption as an additional driver, plus 10 gallons.

b Living in a rural environment increases gasoline consumption by 2 times the combined effect of an additional driver and an additional car.

c The combined hypothesis of (a) and (b)

d A \$0.1 *decrease* in gas price increases consumption by the same amount as an additional car.

Part (b), 8 points

For the sake of continuing our empirical storyline, now assume the dependent variable, gasoline consumption, is given in logged form. As you may recall, in this case the correct marginal effect of a binary (0/1) regressor, such as RURAL above, is given as:

$$\hat{\beta}_p = exp\left(\hat{\beta}\right) - 1$$

where $\hat{\beta}$ is the original estimated coefficient on the binary regressor, and $\hat{\beta}_p$ is the estimated marginal effect, that is the *proportional* change in the dependent variable due to switching the binary regressor from 0 to 1.

a Using the Delta method, show the expression for the asymptotic standard error of $\hat{\beta}_p$. You can label the estimated variance of $\hat{\beta}$ as $\hat{V}\left(\hat{\beta}\right)$.

b Now assume you want to express the correct effect in percentage terms, i.e. you are interested in $\hat{\beta}_{pc} = 100\left(exp\left(\hat{\beta}\right) - 1\right)$. Derive the expression for the asymptotic standard error for $\hat{\beta}_{pc}$.

c Assume the original model was estimated via OLS. For single-variable hypothesis tests regarding either one of these correct effects, which distribution would you use to determine threshold values for rejection? t or z Why?

Part (c), 4 points

Let's return back to the linear-linear model from part (A). Assume now that you are concerned that PRICE may be affected by *measurement error*, since the average regional price is not the one households actually pay at their preferred gas station(s). You decide to instrument this variable with two other variables: (i) average price outside the region (POUT), and (ii) average price of crude oil that week (CRUDE), and switch to a TSLS framework.

You proceed in the typical 2 -stages:

a Under what conditions will these two new variables be valid instruments for PRICE?

b Outline, in general terms, the two steps of this procedure. You can use words or math.

c Which two asymptotically equivalent tests are available to examine if the instrumental variables approach was in fact necessary compared to basic OLS? Why would we generally prefer OLS if possible?

d Assuming you go ahead with these two instruments, show the contents of **Z** - you can use the variable acronyms, no need for math.

Question 3 (suggested time: 20 minutes)

You are involved in a research project on State Park (SP) visitation in West Virginia (WV). Park visitors can be divided into three groups: (1) Day visitors, (2) overnight campers, and (3) overnight hotel or cabin guests. You are interested in the true proportions of these groups for all visitors to *Pipestem*, the "flagship" of WV SPs, on a specific day. Let these true proportions be labeled as π_1, π_2, and π_3, for day visitors, campers, and hotel guests respectively. Naturally, $\sum_{j=1}^{3} \pi_j = 1, \quad j = 1 \ldots 3$. Also, let $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}'$.

On your day of interest, you randomly sample n visitors to Pipestem. For each, you write down an indicator vector $\mathbf{z}_i$ that shows to which group the person belongs. For example, if the person is a day visitor, then $\mathbf{z}_i = \begin{bmatrix} z_{1i} & z_{2i} & z_{3i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Similarly, if the person is a camper, the second element in $\mathbf{z}_i$ will be "1" and the other two will be "0," and if the person is a hotel guest, $z_{3i} = 1$, and $z_{1i} = z_{2i} = 0$. Let the total number of individuals sampled for each group be n_1, n_2, and n_3, respectively.

You stipulate that each individual indicator vector $\mathbf{z}_i$ follows a multinomial likelihood, given as:

$$p\left(\mathbf{z}_i|\boldsymbol{\pi}\right) = \left(\frac{1}{\prod_{j=1}^{3} z_{ji}!}\right) \prod_{j=1}^{3} \pi_j^{z_{ji}} = \prod_{j=1}^{3} \pi_j^{z_{ji}} \tag{1}$$

Part (a), 2 points
Write down the likelihood for the entire sample of n observations. You can label the entire set of n indicator vectors as $\mathbf{z}$. Simplify as much as possible.

Part (b), 5 points
As a prior for $\boldsymbol{\pi}$ you choose a Dirichlet distribution. The density and moments for the Dirichlet are given as:

$$\begin{aligned} & p\left(\boldsymbol{\pi}\right) = \left(\frac{\Gamma\left(\tilde{\alpha}\right)}{\prod_{j=1}^{3} \Gamma\left(\alpha_j\right)}\right) \prod_{j=1}^{3} \pi_j^{\alpha_j - 1}, \quad \alpha_j > 0, \ \forall j, \\ & E\left(\pi_j\right) = \frac{\alpha_j}{\tilde{\alpha}}, \quad V\left(\pi_j\right) = \frac{\alpha_j\left(\tilde{\alpha} - \alpha_j\right)}{\tilde{\alpha}^2\left(\tilde{\alpha} + 1\right)}, \quad \text{where} \\ & \tilde{\alpha} = \sum_{j=1}^{3} \alpha_j \end{aligned} \tag{2}$$

a Derive the *kernel* of the posterior distribution $p\left(\boldsymbol{\pi}|\mathbf{y}\right)$, and determine the statistical distribution for the full posterior.

b Show the posterior parameters for this distribution (label them α_j^*, $j = 1 \ldots 3$).

Part (c), 5 points
Assume you have visitor information from *other nearby SPs with similar facilities*, with average proportions for the three visitor groups of 0.1, 0.5, and 0.4. Interpreting these averages as prior expectations, and letting $\alpha_1 = 1$, derive the prior parameters α_2 and α_3, as well as the prior variances for the three shares. Show the variances with four decimals.

Part (d), 8 points

Assume your Pipestem sample of 200 visitors produces $n_1 = 30$, $n_2 = 100$, and $n_3 = 70$.

a Using all the information from above, compute the posterior expectations and variances for the population shares. Round all expectations to three decimals, and all variances to four decimals.

b How can you tell that the collected data has brought information to the prior?

c The WV State Park Service is willing to sponsor an advertising campaign targeted to *day visitors* (group 1), if the expected posterior share of this group falls below 15%. What will be the town's decision?

Macroeconomics Qualifier Examination

August 2022

Time allocated: 120 minutes.

Question 1:

Consider a two-period overlapping generation model with a population growth rate n. An individual born at time t is endowed with one unit of labor which is supplied inelastically to the market for competitively determined wage rate, w_t. At time t, young individuals make consumption-savings decision. Savings earn a competitively determined return, r_{t+1}. The savings and the returns (on savings) are consumed at the end of period $t+1$. The utility of an individual born at t is given by $U_t = \log c_{1t} + log c_{2t+1}$, where c_{1t} and c_{2t+1} represent young and old age consumptions. The output production technology is given by $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$. Assume that capital depreciates completely during the production process. As a result, savings of the young at time t is the only source of capital during time $t+1$.

(a) Set up the optimization problem of a young individual born at time t and solve for optimal consumption and savings.
(b) Find the expression for the steady-state capital-labor ratio, k^* ,for this economy.
(c) Define k_G^* as the steady-state capital-labor ratio which maximizes steady-state per-capita consumption. Find the value of k_G^*.
(d) Is $k^* = k_G^*$? If not, then consider the following scenario. Suppose that the government using an immigration policy can choose the value of n. What value of n will make $k^* = k_G^*$?

Question 2:

Consider an economy in which each member of a representative household supplies one unit of labor inelastically to the market to earn a wage rate, w_t. Households also lend their asset to earn rental rates, r_t. Both labor and capital markets are competitive and both market clear at each period. Each household maximizes his/her lifetime utility: $\int_0^{\infty} u(c_t)\, e^{-(\rho-n)t} dt$ subject to $\dot{a}(t) = w(t) + r(t)a(t) - c(t) - na(t)$, plus the no-Ponzi game condition. The instantaneous utility is given by, $u(c_t) = \frac{c^{(1-\theta)}-1}{1-\theta}$; population growth rate is represented by n; and ρ represents the rate using which households discount the future. Also, assume that there is no technological progress. Firms combine labor and capital to produce output according to the production function: $Y = AK^{\alpha}L^{1-\alpha}$.

(a) Find the optimal growth rate of consumption.
(b) Derive the expressions for the wage rate, w_t, and the interest rate, r_t. Suppose that it is possible to fix capital-labor ratio at a certain value for the entire duration of the economy. What value of the capital-labor ratio will lead households to consume the same amount every period?
(c) Show that the above maximization problem is equivalent to a household maximizing $\int_0^{\infty} u(c_t) e^{-(\rho-n)t}\, dt$ subject to $\int_0^{\infty} e^{-(r_t-n)t} c_t dt = a_0 + \int_0^{\infty} e^{-(r_t-n)t} w_t dt$. What is the interpretation of this modified budget constraint?
(d) Suppose the government imposes a lump sum tax T on households' asset income. What effect will it have on the growth rate of consumption, the steady-state capital-labor ratio, and the steady-state per-capita consumption?

(e) Suppose that the government taxes households' asset income at a proportional rate τ. What effect will it have on the growth rate of consumption, the steady-state capital-labor ratio, and the steady-state per-capita consumption?