<u>Problem 1</u> (70 minutes)
A domestic firm (Firm 1) and a foreign firm (Firm 2) produce an identical product which they sell in a third country. (The third country assumption allows us to ignore consumer welfare in the computations.)

The government in the domestic country understands the structure of the industry and offers the domestic firm an export subsidy in advance of the quantity decisions by the two firms. Likewise, the government in the foreign country understands the structure of the industry and offers the foreign firm an export subsidy in advance of the quantity decisions by the two firms.

There are two stages in the game:

`Stage 1:` The domestic government sets a per unit export subsidy, s, to maximize the domestic firm's profit net of its total subsidy cost. The foreign government sets a per unit export subsidy, t, to maximize the foreign firm's profit net of its total subsidy cost.

`Stage 2:` The domestic and foreign firms observe the per unit export subsidies set by the two governments. The two firms then play a Cournot game in a third country where they face the homogeneous inverse market demand:

$$P = a - q_1 - q_2$$

It is assumed for simplicity that the two firms have zero costs of production. The payoffs to the domestic and foreign firm are respectively:

$$\pi_1 = Pq_1 + sq_1,$$
$$\pi_2 = Pq_2 + tq_2.$$

The objective of the domestic government is to choose the per unit export subsidy, s, to maximize its welfare function, which is defined as the profits of the domestic firm minus the cost of providing the subsidy:

$$W_1 = \pi_1 - sq_1 = Pq_1.$$

The objective of the foreign government is to choose the per unit export subsidy, t, to maximize its welfare function, which is defined as the profits of the foreign firm minus the cost of providing the subsidy:

$$W_2 = \pi_2 - tq_2 = Pq_2.$$

(a) Determine the subgame perfect equilibrium of the two-stage game. Specify the firms' equilibrium strategies and determine equilibrium subsidies and quantities.

(b) Compare the equilibrium quantities obtained in (a) with the Cournot equilibrium quantities when there are no subsidies. You obtain the latter by setting $s = t = 0$. When is each country's welfare higher?

(c) Suppose that only the domestic government offers a subsidy to its firm, that is, $t = 0$.

- What is the per unit export subsidy set by the domestic government in the subgame perfect equilibrium?
- What are the outputs produced by the two firms in the subgame perfect equilibrium?
- How do these outputs compare with the equilibrium outputs in a Stackelberg game with zero subsidy, firm 1 as leader and firm 2 as follower?

Problem 2 (50 minutes)
Dorothee has a utility-of-wealth function $U(w) = \sqrt{w}$ (called Bernoulli utility function in the textbook and von Neumann-Morgenstern utility function by others) and a current wealth of 2.

A. Will Dorothee invest in a scheme that

- requires an initial capital outlay of 2;
- with probability of 2/3 returns nothing (i.e., the initial outlay is lost and there is no revenue);
- with probability of 1/3 has return 10.5 (i.e., the initial outlay is lost, but there is revenue of 10.5)?

Explain your answer!

B. Betty is identical to Dorothee in every respect. If Dorothee and Betty can share the investment, will they do it? In this case sharing means that each puts up the amount 1 and they split the proceeds of the investment equally. Note: $\sqrt{6.25} = 2.5$.

C. More generally, consider Anna who has initial wealth 2 as well. She has a twice differentiable utility-of-wealth function $V(w)$ that satisfies $V' > 0$ and $V'' < 0$. She can invest a fraction $\alpha \in [0, 1]$ of her initial wealth in the risky project. This investment has revenue

- zero with probability 2/3;
- 10.5α with probability 1/3.

In addition, Anna keeps the amount $2(1-\alpha)$, the part of her initial wealth she does not invest.

(a) Is Anna risk-averse? Explain!

(b) Let $E(\alpha) = \frac{2}{3}V(2(1-\alpha)) + \frac{1}{3}V((1-\alpha)2 + 10.5\alpha)$,
Anna's expected utility as a function of the chosen α.
Show that $E'(\alpha)$ is decreasing in α.

(c) Show that Anna is willing to assume some risk, that is, her optimal choice is an $\alpha^* > 0$. (Check $E'(0)$. Why?)

(d) When does Anna invest her entire initial wealth? (Check $E'(1)$. Why?)

Problem 3 (80 minutes)
Consider a pure exchange economy $\mathcal{E} = \{(X^i, \succsim^i, \omega^i)_{i=1}^2\}$ with two consumers $i \in \{1,2\}$, two commodities $l \in \{1,2\}$ and the initial endowments: $\omega^1 = (0,10)$ and $\omega^2 = (10,0)$. Consumer i has a preference relation $\succsim^i$ over commodity bundles in $X^i = \mathbb{R}^2_+$, that is represented by the following utility function:

$$u^i(x_1, x_2) = \min\left\{x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}; x_1^{\frac{3}{4}} x_2^{\frac{1}{4}}\right\}.$$

Let $p = (p_1, p_2) \in \mathbb{R}^2_+$, $p \neq 0$, denote a price vector. Denote by $m^i \in \mathbb{R}_+$ consumer i's income where $m^i = p\omega^i + t^i$ and by $t^i \in \mathbb{R}$ a transfer payment to i.

1. Derive the consumer i' marginal rate of substitution $MRS^i_{(x_1,x_2)}$ at bundle (x_1, x_2).

2. Derive the demand correspondence $x^i(p, m^i)$ for any $(p, m^i) \in \mathbb{R}^2_+ \times \mathbb{R}_{++}$.

3. Suppose $m^i = p\omega^i$, i.e., there are no transfers $t^i = 0$. In this case, is the demand correspondence homogeneous of degree 0? (Prove or disprove your statement.)

4. Draw an Edgeworth box for $\mathcal{E}$. Determine (formally) and indicate the set all of core allocations.

5. Define a competitive Walrasian Equilibrium (as precise as possible). Derive a competitive WE for the economy $\mathcal{E}$. Is it unique? If not, then determine all equilibria.

6. Determine the set of core allocations that can be generated by a Price Equilibrium with Transfers. I.e., for each core allocation $\tilde{x}$, determine the prices $p = (p_1, p_2)$, income distribution $m = (m^1, m^2)$ and transfers $t = (t^1, t^2)$ that constitute a PET $(\tilde{x}, p, m)$.

Problem 4 (40 minutes)
A finite number k of individuals $i = 1, \dots, k$ make a joint effort to obtain a profit $x \geq 0$ which is equally shared by the members of the group. The effort level of individual i is measured by a number $s_i \in [0,1] \equiv S_i$. The profit is determined by a yield function,

$\pi : S_1 \times \dots \times S_k \to \mathbb{R}_+$,

i.e., $x = \pi(s_1, \dots, s_k)$ is the profit resulting from the joint effort $s = (s_1, \dots, s_k)$.

Each individual i has a utility function $v_i : S_i \times \mathbb{R}_+$ which represents i's preferences over effort-income pairs (s_i, x_i).

The joint decision problem is adequately described by the game

$$\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$$

where

$I = \{1, 2, \dots, k\}$,
$S_i = [0,1]$ for $i \in I$,
$u_i(s_1, \dots, s_k) = v_i(s_i, \frac{1}{k}\pi(s_1, \dots, s_k))$ for all i, $s = (s_1, \dots, s_k) \in S$.

DETERMINE the Nash equilibria (in pure strategies) for the special case

$\pi(s_1, \dots, s_k) = \sqrt{\sum_{i=1}^{k} s_i^2}$,
$v_i(s_i, x_i) = x_i - \frac{1}{4}s_i^4$,
$u_i(s_1, \dots, s_k) = \frac{1}{k}\sqrt{\sum_{j=1}^{k} s_j^2} - \frac{1}{4}s_i^4$.

Question 1 (suggested time: **20** minutes).
Consider the simple (one parameter) Normal model in table 1.

Table 1 - The simple (one parameter) **Normal model**		
Statistical GM:		$X_t = \mu + u_t,\ t\in\mathbb{N},$
[1]	Normal:	$X_t \backsim \mathsf{N}(.,.)$, i.e. $f(x;\mu)=\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\},\ \mu\in\mathbb{R},\ x\in\mathbb{R},$
[2]	Constant mean:	$E(X_t)=\mu$, for all $t\in\mathbb{N}$,
[3]	Constant variance:	$Var(X_t)=\sigma^2$ (known),
[4]	Independence:	$\{X_t,\ t\in\mathbb{N}\}$ is an independent process.

(A) (a) Explain why $\widehat{\mu}(\mathbf{X})=\overline{X}_n=\frac{1}{n}\sum_{k=1}^{n} X_k$ is the Maximum Likelihood estimator of μ and state its finite sample optimal properties in terms of its sampling distribution:

$$\frac{\sqrt{n}(\overline{X}_n-\mu)}{\sigma} \backsim \mathsf{N}(0,1). \tag{1}$$

(b) Explain why the result in (1) is ambivalent because the underlying reasoning is not stated explicitly. After adding the latter, derive a 2-sided $(1-\alpha)$ coverage probability Confidence Interval (CI) for μ^*.

(c) Using your answers in (a)-(b) explain why the following inferential claims are unwarranted:

$$\begin{aligned} &\text{(i) } \widehat{\mu}(\mathbf{x}_0) \simeq \mu^* \text{ for a large enough } n,\\ &\text{(ii) } \mu^*\in[L(\mathbf{x}_0), U(\mathbf{x}_0)], \end{aligned} \tag{2}$$

where $\widehat{\mu}(\mathbf{x}_0)=\overline{x}_n$ and $L(\mathbf{x}_0), U(\mathbf{x}_0)$ denote the observed lower and upper bounds.

(B) (a) Specify explicitly the optimal α-significance level Neyman-Pearson (N-P) test T_α for the hypotheses:

$$H_0\text{: } \mu \leq \mu_0, \text{ vs. } H_1\text{: } \mu > \mu_0, \tag{3}$$

and (b) define and explain its optimal properties: (i) Uniformly Most Powerful and (ii) Consistent.

(c) Using your answers in (a)-(b) explain why the following inferential claims are unwarranted:

$$\begin{aligned} &\text{(i) Accept } H_0 \text{ entails that there is evidence for } H_0,\\ &\text{(ii) Reject } H_0 \text{ entails that there is evidence for } H_1, \end{aligned} \tag{4}$$

by relating (i)-(ii) to the fallacies of acceptance and rejection, respectively.

(C) Explain briefly how the post-data severity evaluation of the testing results (accept/reject H_0) can addresses the two fallacies in (4) by outputting the warranted discrepancy γ_1 from $\mu=\mu_0$, (i.e. $\mu_1=\mu_0+\gamma_1$) for the particular data $\mathbf{x}_0$.

NOTE that a hypothesis H passes a *severe test* T_α with data $\mathbf{x}_0$ if:

(S-1) $\mathbf{x}_0$ accords with H, and (S-2) with very high probability, test T_α would have produced a result that accords less well with H than $\mathbf{x}_0$ does, if H were false.

Question 2
(suggested time: 20 minutes)

Consider the following CLRM for a sample of n observations:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where the error term has the usual CLRM properties.

Assume the dependent variable measures the cost of a one-hour power outage to a given household, in dollars. You can alternatively interpret this as the household's "willingness to pay" (WTP) to avoid the outage.

The matrix $\mathbf{X}$ of explanatory variables contains the following:

1. C: vector of ones (intercept term, as usual)
2. BUS: binary indicator (1= households runs a business from home, 0 = otherwise)
3. ELD: number of occupants over 64 years of age
4. APP: number of electronic appliances in the residence
5. CHI: number of children age 6 or younger

The corresponding coefficients are labeled β_1 through β_5.

Part (a)
For the following linear hypotheses, to be tested via an F-test, show the form of $\mathbf{R}$ and $\mathbf{q}$ when the hypothesis is written in the form of $H_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{q}$, where $\boldsymbol{\beta}$ is the full vector of coefficients: (note: $\mathbf{q}$ can be a vector or a scalar, so the bold-facing is just a general representation).

a The presence of an additional child increases outage cost by two times more than the presence of an additional appliance.

b An additional child increases cost by the same as an additional elderly.

c The combined hypothesis of (a) and (b)

d Running a business from home increases costs by the same as an additional child PLUS an additional elderly

Part (b)
No suppose you want to use a Wald test to examine the following hypothesis:

$$H_0 : \; ln\left(\frac{\beta_2}{\beta_3}\right) = \left(ln\left(\frac{\beta_2}{\beta_5}\right)\right)^2$$

As usual, you set up the test in the form $c(\boldsymbol{\beta}) = q$, where $\boldsymbol{\beta}$ is again the full vector of coefficients, and $c(.)$ is the nonlinear function on the elements of $\boldsymbol{\beta}$ implied by the H_0.

a Letting $q = 0$, show the exact form of $c(\boldsymbol{\beta})$

b Show the exact form of derivative vector or matrix $\mathbf{C}$, where $\mathbf{C} = \frac{\partial c(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}$

c Assume the threshold value for rejection at a 5% level of significance for this test equals 3.84, and your test statistic computes at 7.51. What is your decision regarding the H_0?

d How many parameter restrictions are underlying this test?

Part (c)
Assume now that you are concerned that the number of appliances (APP) may be an endogenous regressor, since the over- or mis-use of certain appliances can itself lead to outages. You decide to instrument this variable with two other variables: (i) years family has lived in that home (YRS), and (ii) the number of outages experienced in the preceding 12 months (NUM), and switch to a TSLS framework.

You proceed in the typical 2 -stages:

Stage 1: Compute $\hat{\mathbf{X}} = \mathbf{P_Z X} = \mathbf{Z}\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\mathbf{X}$
Stage 2: compute the TSLS estimator as $\mathbf{b}_{TSLS} = \left(\hat{\mathbf{X}}'\hat{\mathbf{X}}\right)^{-1}\hat{\mathbf{X}}'\mathbf{y}$

a Under what conditions will / would these two new variables be valid instruments for APP?

b What might be a problem with validity for one or both instruments in this case? A brief verbal discussion will do.

c Assuming you go ahead with these two instruments, show the contents of $\mathbf{Z}$ - you can use the variable acronyms, no need for math.

Question 3
(suggested time: 20 minutes)

You are considering opening a local shoe store in a rural area where the main competition comes from online shopping. Your competitive angle is that your store will likely offer a higher probability that a chosen pair of shoes will fit for a given customer, in contrast to online shopping which often requires several rounds of returns and re-tries before a fitting pair is found.

You poll a sample of n recent online shoe shoppers in your market area and capture the number of times they *had to return an order* before they found a fitting pair. Let that *number of returns* be denoted as y_i for customer i. To make things perfectly clear, if it took 3 rounds to find a correct pair, $y_i = 2$.

You look at your data and decide that it is well characterized by a geometric distribution, given as:

$$\begin{aligned} &p\left(y_i|\theta\right) = \theta * \left(1-\theta\right)^{y_i}, \\ &y = 0, 1, 2, 3, \ldots, \quad 0 < \theta < 1 \end{aligned} \tag{1}$$

Here, parameter θ is the (constant) probability that a pair of shoes fit at any given trial (or order).

You further decide to estimate θ via a Bayesian approach. Since it is a probability bounded by 0 and 1, you opt for a Beta prior, given as:

$$\begin{aligned} &p\left(\theta\right) = \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\theta^{\alpha-1}\left(1-\theta\right)^{\beta-1}, \quad \text{with} \\ &\alpha, \beta > 0, \\ &E\left(\theta\right) = \frac{\alpha}{\alpha+\beta} \\ &V\left(\theta\right) = E\left(\theta\right)\frac{\beta}{\left(\alpha+\beta\right)\left(\alpha+\beta+1\right)}, \end{aligned} \tag{2}$$

where α and β are shape parameters.

Part (a), 12 points

a Write out the likelihood function for your sample of n observations. Simplify as much as possible.

b Derive the posterior kernel and show that it also characterizes a Beta distribution. Derive the posterior shape parameters α^* and β^*.

c Compare the prior and posterior expectation of θ. Derive the condition for $\sum_{i=1}^{n} y_i$ in terms α, β, and n for the posterior expectation to be larger than the prior expectation (in you case: higher probability that a pair of shoes fit at any given trial).

Part (b), 8 points

Your actual data produces reports from 100 customers, with a combined total number of returns of 151. Furthermore, data from other market areas suggest an average return rate of 0.5. You use this information to form a Beta prior with shapes $\alpha = \beta = 2$.

a Derive the posterior shapes, and the posterior expectation of θ. How does it compare with your prior? Given your derivation above, how small would the number of returns have to be to make the posterior expectation larger then the prior expectation?

b Derive the prior and posterior variance of θ. Be precise to the fourth decimal. Has your collected data brought useful information to the problem? How can you tell?

c You decide that your store would be competitive if you can beat the posterior probability of "fit" by at least 20%, on average (by careful shoe selection and better customer service and in-store advising). How large would your average fitting probability have to be?

Macroeconomics Qualifier Examination

June 2022

Time allocated: 120 minutes.

Question 1:

Consider an economy consisting of a constant population of infinitely lived individuals. A representative household seeks to maximize lifetime utility $U = \int_0^\infty u(c_t)\, e^{-\rho t} dt$, where ρ represents the discount factor. The instantaneous utility function is given by $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$. Each household combines its fixed labor endowment and capital to produce output according to the production function: $Y = AK^\alpha L^{1-\alpha}$. The government taxes output at the rate τ_Y, taxes labor supply at the rate τ_L (a lump-sum tax), provides per-capita lump-sum transfers in the amount φ, and purchases goods and services in the per capita amount g. Thus, in the per capita terms, the budget constraint facing a household is given by $\dot{k}_t = (1-\tau_Y)Ak_t^\alpha - \tau_L - c_t - \delta k_t + \varphi$, where δ represents the depreciation rate on capital. Assume that the government expenditure, g, neither provides utility nor improves productivity. Also, assume that the government balances the budget, and in the per capita terms, its budget constraint is given by $g_t = \tau_Y Ak_t^\alpha + \tau_L - \varphi$.

(a) What are the household's first-order optimization conditions (FOCs), assuming that the representative household takes τ_Y, τ_L, φ, and g as given?

(b) What is the steady-state level of k in this economy?

(c) Suppose that the government surprises households by permanently raising the values of τ_Y and g (without changing τ_L and φ). Use a phase-diagram in (k, c) space to show what happens to the steady-state values of k and c.

(d) Suppose that the government raises τ_L and g (without changing τ_Y or φ). What happens to the steady-state values of k and c? What makes the results different from the previous results?

(e) Suppose that the government raises τ_L and φ (without changing τ_Y or g). What happens to the steady-state values of k and c? Explain why the effects of this policy change are different from those in parts (c) and (d).

Question 2:

Consider an economy with zero population growth. A central planner makes the decision on behalf of representative agents and wishes to maximize $\sum_{t=0}^\infty \beta^t \frac{c_t^\gamma}{\gamma}$, where c_t is the consumption per capita, β represents the discount factor, and $\gamma < 1$. Each member of the population is endowed with one unit of labor. The central planner makes the decision about how to allocate agents' labor endowment between the production of output and the accumulation of human capital, h. Each individual can produce output using the technology $y_t = \alpha h_t u_t$, where h_t is the amount of human capital available to an individual. The fraction of labor endowment allocated to output production is denoted by u_t, and $\alpha > 0$ represents a technological parameter. Human capital is produced using the technology $h_{t+1} = \delta h_t(1-u_t)$, where $\delta > 0$ and $(1-u_t)$ denotes the fraction of time t labor endowment allocated toward the accumulation of human capital. Assume that the initial stock of human capital, h_0, is given. Finally, assume that each individual consumes the entire output, i.e., $c_t = y_t = \alpha h_t u_t$.

a. Set up the optimization problem for the central planner.
b. Use Bellman Equation to solve the problem and derive the Euler equation for consumption.
c. Derive the growth rate of consumption, $\frac{c_{t+1}}{c_t}$.
d. How u_{t+1} and u_t are related along the optimal path?
e. Use your answer in part (d) to derive the steady-state value of u and denote it as u^*. Comment on the long-run outcome for this economy if $u_0 \neq u^*$
f. Suppose that $u_0 = u^*$. Use this condition to derive the growth rate of human capital, $\frac{h_{t+1}}{h_t}$. What is the relationship between $\frac{c_{t+1}}{c_t}$ and $\frac{h_{t+1}}{h_t}$?