Problem 1 (30 minutes)

The market for a certain product consists of two firms, firm A and firm B. The inverse demand for the product is given by

$$p = 100 - q$$

where p is price and q is total output, i.e. $q = q_A + q_B$. The total cost functions of the two firms are

$$TC_A(q_A) = 20q_A$$

$$TC_B(q_B) = 25q_B.$$

Also, the government imposes in this market a ceiling price of $\widehat{p} = 30$ so that no firm can sell its output at a higher price.

(a) Derive the price in the market and the quantity produced by each firm in a Cournot equilibrium.

(b) Suppose now that there are n firms in the market, where n is an even number greater than 2, and that for any n half the firms are of type A and the other half are of type B. How will the price in an equilibrium change, if at all, as n increases? What is the minimum that a price in an equilibrium will ever reach?

Problem 2 (60 minutes)

Throughout, the following assumptions are maintained for the economy of Marvelous Island. There are three goods: good one is an input with price $p_1 = 1$, whereas goods two and three are outputs, with prices $p_2 > 0$ and $p_3 > 0$ respectively. All economic agents are price takers.

PART I: PRODUCTION. There is one firm with technology (production set)

$$Y = \{(y = (y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 \leq 0, \ \max\{y_2, y_3\} + y_1 \leq 0\}$$

That is, $\mathcal{F}(y_1, y_2, y_3) = \max\{y_2, y_3\} + y_1$ is the essential part of the firm's transformation function.

(a) For which price systems $\mathbf{p}= (1,\ p_2, p_3)$ with $p_2 > 0$, $p_3 > 0$ does the profit maximization problem

$$\max\ \mathbf{p}y \text{ subject to } y \in Y$$

have a solution? (It might be helpful to determine the type of returns to scale exhibited by the technology Y).

(b) In case the profit maximization problem has a solution, determine the maximum profit $\pi(\mathbf{p})$ and the set of profit maximizing production plans, $y(\mathbf{p})$.

Part II: GENERAL EQUILIBRIUM. At prices $p_1 = 1$, $p_2 > 0$ and $p_3 > 0$, the aggregate demand is A_2/p_2 for good 2 and A_3/p_3 for good 3, where A_2 and A_3 are positive constants. The aggregate endowment of consumers is $\omega = (\omega_1, 0, 0)$ with $\omega_1 > 0$.

(c) Determine $p_2^* > 0$ and $p_3^* > 0$ at which markets are cleared, using the findings in **PART I.**

PART III: CONSUMERS. Suppose all consumers have identical Cobb-Douglas preferences. Each spends 40% of his income on the consumption of good 1.

(d) Provide a utility function for these consumers consistent with the given information.

(e) Determine ω_1, using the findings of **PART I** and **PART II.**

Problem 4 (30 minutes)

A consumer has the expenditure function

$$e(u, p) = \overline{u} \min\{p_1, p_2\}.$$

Derive his:

(a) Indirect utility function,

(b) Marshallian (regular) demand functions,

(c) Hicksian (compensated) demand functions,

(d) Slutsky Equation.

Problem 3 (60 minutes)

Consider the following cases. Assume throughout that the firms are risk-neutral.

CASE 1.

Tomorrow, with equal probability, it will either rain or shine. Firms 1 and 2 have to make their production plans today not knowing what the weather will be like tomorrow. They simultaneously choose whether to produce umbrellas or sun lotion. Their profits are as follows: if they make the same choice (both choose umbrellas or both choose lotion) then each makes a profit of 1 regardless of the actual weather. However if they make different choices then the firm that makes the choice appropriate to the actual weather makes a profit of 4, while the other one makes a profit of 0 (e.g. if firm 1 chooses lotion, firm 2 chooses umbrellas and the weather turns out to be rain, then firm 1 makes a profit of 0 and firm 2 makes a profit of 4.

(a) Draw the extensive form of the game and derive all Subgame Perfect Nash Equilibria.

(b) Write the corresponding normal-form game and derive all the pure-strategy Nash Equilibria.

CASE 2.

This case is a modification of case 1. In this case firm 1 chooses first (not knowing what the weather will be like). Then, after learning what firm 1 actually chose, but still not knowing what the weather will be like, firm 2 makes its choice.

(c) Draw the extensive form of the game and derive all Subgame Perfect Nash Equilibria.

(d) Write the corresponding normal-form game and derive all the pure-strategy Nash Equilibria.

1. **A**. The traditional textbook approach to the omitted variables' bias problem argues its case as follows.

The estimated linear regression model takes the form:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \quad \mathbf{u} \backsim \mathsf{NIID}(\mathbf{0}, \sigma_u^2 \mathbf{I}_T),\ t \in \mathbb{N} := \{1, 2, ..., \}$$

yielding OLS estimators:

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}, \quad s^2 = \tfrac{1}{T-k}\widehat{\mathbf{u}}'\widehat{\mathbf{u}}, \quad \widehat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}.$$

However, the 'true' linear regression model is:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} \backsim \mathsf{NIID}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T),\ t \in \mathbb{N}. \tag{1}$$

Hence, the OLS estimators are, in general:

$$\begin{array}{lll} \text{biased:} & E(\widehat{\boldsymbol{\beta}}) \neq \boldsymbol{\beta}, & E(s^2) \neq \sigma_\varepsilon^2, \\ \text{inconsistent:} & p\lim(\widehat{\boldsymbol{\beta}}) \neq \boldsymbol{\beta}, & p\lim(s^2) \neq \sigma_\varepsilon^2. \end{array}$$

(a) Explain the above argument and bring out any weaknesses.

(b) Do you agree or disagree with the conclusions of this argument. Explain your answer.

B. (a) For the linear regression model as specified above (without the Normality assumption), state and explain the Gauss-Markov theorem.

(b) Use the Gauss-Markov theorem to construct a test for the following hypotheses:

$$H_0 : \boldsymbol{\gamma} = \mathbf{0} \text{ vs. } H_1 : \boldsymbol{\gamma} \neq \mathbf{0},$$

in the context of (1). Hint: use only the results of the Gauss-Markov theorem without the Normality assumption.

Qualifying Exam Question

Instructions: This question requires excellent use of notation. Please pay attention to all of your technical arguments.

Let y_i be a discrete variable taking the values 0 or 1. Let x_i be a $1 \times k$ vector of covariates on $\mathbb{R}^k$, ε_i is a random error, and β is a $k \times 1$ vector of response coefficients. We define the following probabilities over the random variables $x_i\beta \in \mathbb{R}$:

(a) $Pr(x_i\beta > 1) = \pi$, $Pr(x_i\beta \in [0,1]) = \gamma$, $Pr(x_i\beta < 0) = \rho$. Here $\pi + \gamma + \rho = 1$.

(b) $\kappa_\gamma = \{i|x_i\beta \in [0,1]\}$, $\kappa_\pi = \{i|x_i\beta > 1\}$.

1. Show that

   (i) $Pr(i \in \kappa_\pi) = \pi$.

   (ii) $Pr(i \in \kappa_\gamma) = \gamma$.

   (iii) $Pr(i \notin \kappa_\pi \cup \kappa_\gamma) = \rho$.

2. Our data generating process (DGP) is: $y_i = 1$ for $i \in k_\pi$, $y_i = x_i\beta + \varepsilon_i$ for $i \in k_\gamma$, and $y_i = 0$ for $i \notin \kappa_\pi \cup \kappa_\gamma$). Show that the conditional probabilities for y_i are:

$$\begin{aligned} Pr(y_i = 1|x_i,\ i \in \kappa_\pi) &= 1, \\ Pr(y_i = 1|x_i,\ i \in \kappa_\gamma) &= x_i\beta, \\ Pr(y_i = 0|x_i,\ i \in \kappa_\gamma) &= 1 - x_i\beta, \\ Pr(y_i = 0|x_i,\ i \notin \kappa_\pi \cup \kappa_\gamma) &= 1, \end{aligned}$$

   and the conditional probabilities for the error process are

$$\begin{aligned} Pr(\varepsilon_i = 0|x_i,\ i \in \kappa_\pi) &= 1, \\ Pr(\varepsilon_i = 1 - x_i\beta|x_i,\ i \in \kappa_\gamma) &= x_i\beta, \\ Pr(\varepsilon_i = -x_i\beta|x_i,\ i \in \kappa_\gamma) &= 1 - x_i\beta, \\ Pr(\varepsilon_i = 0|x_i,\ i \notin \kappa_\pi \cup \kappa_\gamma) &= 1. \end{aligned}$$

3. If we wanted to estimate our DGP using OLS we proceed as follows:

$$y_i = x_i\beta + u_i, \text{ for } i = 1, \ldots, n, \tag{1}$$

   where u_i is a zero mean random variable and is independent of x_i. **Note that the OLS error term, u_i is not equivalent to ε_i.** The OLS error term, u_i is defined as $u_i = 1 - x_i\beta$ for $i \in \kappa_\pi$, $u_i = y_i - x_i\beta$ for $i \in \kappa_\gamma$ and $u_i = -x_i\beta$ for $i \notin \kappa_\pi \cup \kappa_\gamma$. Show that the conditional probabilities of u_i are

$$\begin{aligned}
Pr(u_i = 1 - x_i\beta | x_i, \ i \in \kappa_\pi) &= 1, \\
Pr(u_i = 1 - x_i\beta | x_i, \ i \in \kappa_\gamma) &= x_i\beta, \\
Pr(u_i = -x_i\beta | x_i, \ i \in \kappa_\gamma) &= 1 - x_i\beta, \\
Pr(u_i = -x_i\beta | x_i, \ i \notin \kappa_\pi \cup \kappa_\gamma) &= 1.
\end{aligned}$$

4. Please show that OLS estimation of the DGP described above is biased and inconsistent when $\gamma < 1$. **Hint:** show that the conditional expectation of the OLS error term, u_i can be written as a function of x_i.

5. Provide a short essay on why it is intuitive that OLS estimation of this type of DGP is inappropriate and draw a picture for the one dimensional case for what is occurring.

Problem 3. (75 minutes)

Consider the Ramsey model with no technological change, and zero population growth. An individual's output is $y_t = 2Ak_t^{1/2}$ and his budget constraint is

[1] $\dot{k}_t = [2Ak_t^{1/2} - \delta k_t - c_t]$,

in which k is capital, c is consumption, and the dot notation refers to the variable's time derivative. The individual seeks to solve

$$\max \int_0^\infty u[c(s)]\exp(-s\theta)ds$$, s.t. [1], given k_0 and $u(c) \equiv 2c^{1/2}$.

(a) Show that the optimal path for consumption is

[2] $\dfrac{\dot{c}_t}{c_t} = 2[Ak_t^{-1/2} - \delta - \theta]$.

(b) Find the steady state for the capital stock and consumption.

The dynamic equations for the economy are

[1′] $\dot{k}_t / k_t = [2Ak_t^{-1/2} - \delta - c_t / k_t]$ and [2] $\dfrac{\dot{c}_t}{c_t} = 2[Ak_t^{-1/2} - \delta - \theta]$.

To determine the shape of the dynamic path to the steady state, consider the following. Let $z_t \equiv c_t / k_t$.

(c) Show that $\dfrac{\dot{z}}{z} = \dfrac{\dot{c}}{c} - \dfrac{\dot{k}}{k}$ = z_t-(δ +2θ).

(d) In *z*-*k* space with *z* on the vertical axis and *k* on the horizontal, graph the curve $\dot{z}_t$=0. Indicate when $\dot{z}_t$<0, and when $\dot{z}_t$>0.

(e) In *z*-*k* space with *z* on the vertical axis and *k* on the horizontal Graph the curve $\dot{k}_t$=0. Indicate when $\dot{k}_t$<0, and when $\dot{k}_t$>0. The relevant equation: [1′] $\dot{k}_t / k_t = [2Ak_t^{-1/2} - \delta - c_t / k_t] = [2Ak_t^{-1/2} - \delta - z_t]$.

(f) Use the two curves from (d) and (e) to draw a phase diagram in *z-k* space. Indicate the path to the steady state. What happens to the consumption/capital ratio as the economy converges to the steady state. What is the steady state *c/k* ratio? Explain.

(g) Draw the phase diagram for the economy in *c-k* space with k on the horizontal axis.. Indicate the path to the steady state. What is the shape of the path to the steady state? The relevant dynamic equations are

$$\frac{\dot{c}_t}{c_t} = 2[Ak_t^{-1/2} - \delta - \theta] \text{ and } \dot{k}_t = [2Ak_t^{1/2} - \delta k_t - c_t].$$

Consider the effects of a tax on capital. The tax rate on capital is constant at τ. Assume also that individuals receive a lump-sum payment from the government equal to s_t. Net taxes are zero, so τk_t - s_t =0. (Of course, each individual takes τ and s_t as given in making his consumption/saving decision.)

(h) Initially τ =0 and s_t =0. The tax rate increases to $\hat{\tau}$ >0. Find the effects on the steady state capital and the c/k ratio. Explain.

(i) Show the effect of the change in (h) using the phase diagram in *z-k* space. Explain.

(j) Show the effect of the change in (h) using the phase diagram in *c-k* space. Explain.