MICRO QE, MARCH 2015

Problem 1 (Hans, 30 min.)
Consider the case of two goods and a consumer with utility representation

$$U(x_1, x_2) = \begin{cases} 2x_1 + x_2 & \text{for } x_2 \geq x_1 \geq 0; \\ x_1 + 2x_2 & \text{for } 0 \leq x_2 \leq x_1. \end{cases}$$

Let $p_1 > 0, p_2 > 0, m > 0$.
Solve the consumer's utility maximization problem

$$\max\ U(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq m.$$

Problem 2 (Hans, 50 min.)
This problem is about a Bertrand duopoly. There are two firms that can produce "super-steel", namely United Steel Company (USC) and Atlantic Coast Steel (ACS). Super-steel prices are discrete and belong to $\mathbb{P} = \{0, 1, 2, \ldots, 100\}$. The market demand for super-steel is

$100 - P$ for $P \in \{0, 1, 2, \ldots, 100\}$.

Let $P_1 \in \mathbb{P}$ denote the price charged by USC and $P_2 \in \mathbb{P}$ denote the price charged by ACS. If a firm charges a lower price than its competitor, then that firm gets the entire market demand at its price. If both firms charge the same price, then they split the market equally.

$C_1 = 20$ is the constant marginal cost of USC. ACS has an outmoded technology. It can make an investment to improve its technology. There are four possible investment choices:

(i) ACS does not invest anything (chooses $I = 0$) and has constant marginal cost $C_2 = 80$.

(ii) ACS invests $I = 300$ and then has constant marginal cost $C_2 = 40$.

(iii) ACS invests $I = 600$ and then has constant marginal cost $C_2 = 20$.

(iv) ACS invests $I = 900$ and then has constant marginal cost $C_2 = 10$.

Here are your tasks:

(a) Determine the Bertrand equilibria in all four cases (i)–(iv).

(b) Consider the two-stage game where in the first stage, ACS chooses an investment $I \in \{0; 300; 600; 900\}$ and in the second stage there is Bertrand competition.

Determine the subgame perfect equilibria of the two-stage game where ACS's profit equals its duopoly profit minus I!

Problem 3 (Eric, 50 min.)

An owner is considering a first-price, sealed-bid auction for the sale of his property. There are two bidders whose valuations of the property, v_1 and v_2, are independent and identically distributed over $[0, 1]$. Assume that the valuation of every bidder is less than x with probability $F(x) = x^2$, for any $x \in [0, 1]$. Each buyer knows the existence of the other bidder, but the realized valuations are private information. The seller and the two buyers all know the cumulative distribution function F.

(a) Find the symmetric Bayesian Nash equilibrium (BNE).
HINT. Look for a BNE where the players use the same increasing linear strategy: $b_i(v_i) = \alpha v_i$, for any $i \in \{1, 2\}$ and $v_i \in [0, 1]$. You have to find the equilibrium value of α.

(b) Compute the expected revenue of the seller under the BNE of this first-price auction.

(c) Suppose that the seller instead decides to use a second-price, sealed-bid auction.

(c.1) Argue that it is a weakly dominant strategy for each buyer to bid their exact valuation. What is then the BNE of this second-price auction?

(c.2) Compute the seller's expected revenue. Is it higher than under the first-price auction?

Problem 4 (Adam, 50 min.)

Consider a private ownership economy with one consumer, called Robinson, one firm and two commodities. The first commodity is a consumption good. The second commodity is leisure; the difference between the total amount of time that Robinson has, 12 hours, and his working time l. Robinson is endowed with zero units of the consumption good. His preferences $\succsim$ on the consumption space $X = \{(x, l) \in \mathbb{R}^2_+ : 0 \leq l \leq 12, 0 \leq x\}$ are represented by the Cobb-Douglas utility function:

$$u(x, 12 - l) = x \cdot (12 - l).$$

The firm uses labor to produce the consumption good. The firm has the production function:

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq l \leq 4; \\ 2\sqrt{l - 4} & \text{if } l > 4. \end{cases}$$

The perfectly competitive firm is owned by Robinson. Denote by p the price of the consumption good and by w the wage rate; $(p, w) \in \mathbb{R}^2_{++}$.

1) Derive Robinson's demand function $x^R(p, w)$ and supply function $l^R(p, w)$.

2) Draw the technology in a diagram. What returns to scale does the technology display?

3) Derive the firm's supply function $x^F(p, w)$ and its demand function $l^F(p, w)$.

4) Determine the Walrasian equilibrium, provided it exists. Use price normalization $w = 1$, if possible.

5) Does the Second Welfare Theorem hold true in this economy? Please, provide an explanation.

Question 1 (30 minutes)

(a) Consider the simple Normal model:

$$X_t \backsim \text{NIID}(\mu, \sigma^2),\ t{=}1, 2, ..., n, ...,$$

with σ^2 is <u>known</u>; 'NIID' stands for 'Normal, Indepenent and Identically Distributed'.

(i) State the sampling distributions of the test statistic:

$$d(\mathbf{X}){=}\frac{\sqrt{n}(\overline{X}_n-\mu_0)}{\sigma},\quad \overline{X}_n{=}\frac{1}{n}\sum_{t=1}^{n} X_t,$$

under the null and under the alternative hypotheses:

$$H_0\colon\ \mu = \mu_0,\ \text{vs.}\quad H_1\colon\ \mu > \mu_0. \tag{1}$$

(ii) Define the optimal Neyman-Pearson (N-P) test in terms of $d(\mathbf{X})$ and explain what 'optimal' means in this case.

(b) (i) Compare and contrast the sampling distributions of $d(\mathbf{X})$ under H_0 and H_1 to:

$$d(\mathbf{X}; \mu) = \frac{\sqrt{n}(\overline{X}_n-\mu^*)}{\sigma} \overset{\mu=\mu^*}{\backsim} \mathsf{N}(0, 1), \tag{2}$$

where μ^* denotes the 'true' value of μ, whatever that happens to be.

(ii) Explain how (2) can be used to construct an $(1{-}\alpha)$ Confidence Interval for μ.

(c) Define and explain the notions of (i) type I error probability, (ii) type II error probability, (iii) power of the test and (iv) the p-value, and (v) compare and contrast (i) and (iv).

(d) (i) State the fallacies of acceptance and rejection and explain why the accept/reject rules and the p-value are vulnerable to these fallacies when they are interpreted as providing evidence for or against the null or the alternative.

(ii) Relate your answer in (i) to the difference between statistical and substantive significance.

Question 2 (30 min.)

Consider the following regression model:

$$\mathbf{y} = \beta_0 + \beta_1 \mathbf{x} + \epsilon \tag{1}$$

where all vectors are of length n and the β's are, respectively, a scalar intercept and coefficient. The single explanatory variable is continuous, but potentially endogenous. You decide to use a single instrument, $\mathbf{z}$, to obtain an unbiased estimate of β_1.

The instrument $\mathbf{z}$ is binary - n_0 of its elements take the value of 0, and the remaining n_1 elements take the value of 1. For convenience, you can assume that the data are sorted such that the zero elements are listed first.

Further define $\bar{y}$ as the mean of $\mathbf{y}$, $\bar{y}_0$ as the mean of $\mathbf{y}$ for the observations for which $z_i = 0$ (that is, the first n_0 observations), and $\bar{y}_1$ as the mean of $\mathbf{y}$ for the observations for which $z_i = 1$. Similarly define $\bar{x}$, $\bar{x}_0$, and $\bar{x}_1$.

Part (a)

Show that:

1. $n\bar{z} = n_1$
2. $\bar{y} = \frac{n_0\bar{y}_0 + n_1\bar{y}_1}{n}$
3. $\sum_i^n z_i y_i = n_1 \bar{y}_1$
4. $\sum_i^n z_i x_i = n_1 \bar{x}_1$

Part (b)

Write down the generic matrix form of the simple IV estimator for this model (call it $\boldsymbol{\beta}_{IV}$), using $\mathbf{Z} = \begin{bmatrix}\mathbf{i} & \mathbf{z}\end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix}\mathbf{i} & \mathbf{x}\end{bmatrix}$.

Letting $\boldsymbol{\beta}_{IV} = \begin{bmatrix}\hat{\beta}_0 & \hat{\beta}_1\end{bmatrix}'$, show that $\hat{\beta}_1$ can be written as $\hat{\beta}_1 = \frac{\sum_i^n z_i y_i - n\bar{z}\bar{y}}{\sum_i^n z_i x_i - n\bar{z}\bar{x}}$.

Hint: recall that the inverse of a 2 by 2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ is given as $\frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix}$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$.

Part (c)

Using your results from part (a), show that this can be further simplified to $\hat{\beta}_1 = \frac{\bar{y}_1 - \bar{y}_0}{\bar{x}_1 - \bar{x}_0}$.

Part (d)

Can you provide some intuition for this result?

To make things more concrete, lets assume that y denotes wage, x denotes days spent in a training program, and z denotes if a sibling has also taken the training program.

Question 3 (30 min.)

Consider the standard random effects (RE) model:

$$y_{it} = \mathbf{x}_{it}'\boldsymbol{\beta} + u_{it}, \quad \text{where} \\ u_{it} = \mu_i + \nu_{it}, \quad t = 1, \ldots, T. \tag{1}$$

The model satisfies the standard random effects model assumptions:

$$E(\mu_i|\mathbf{x}_{it}) = 0, \quad E(\nu_{it}|\mathbf{x}_{it}) = 0 \quad \forall i, t \\ Var(\mu_i|\mathbf{x}_{it}) = \sigma_\mu^2 \quad Var(\nu_{it}|\mathbf{x}_{it}) = \sigma_\nu^2 \\ Cov(\mu_i, \nu_{it}|\mathbf{x}_{it}) = 0, \tag{2}$$

where "*Var*" denotes "variance." and "*Cov*" means "covariance."

Part (a)
Let $\bar{u}_i$ be the average, over time, of u_{it}. Compute $Var(\bar{u}_i)$ and $Cov(u_{it}, \bar{u}_i)$.

Part (b)
Argue, in words or math, that the following must hold, given the model assumptions:

$$E(u_{it}|\mathbf{x}_{it}, \bar{u}_i) = E(u_{it}|\bar{u}_i) \tag{3}$$

Part (c)
Recall the generic formula for the conditional expectation of two jointly normally distributed random variables x and z: $E(x|z) = E(x) + \frac{Cov(x,z)}{Var(z)}(z - E(z))$. Using your results from part (a), show that

$$E(u_{it}|\bar{u}_i) = \bar{u}_i \tag{4}$$

Part (d)
Use your results from parts (b) and (c) to show that

$$E(y_{it}|\mathbf{x}_{it}, \bar{y}_i) = \mathbf{x}_{it}'\boldsymbol{\beta} + (\bar{y}_i - \bar{\mathbf{x}}_i'\boldsymbol{\beta}), \quad \text{where} \\ \bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}, \\ \bar{\mathbf{x}}_i = \frac{1}{T}\left[\sum_{t=1}^{T} x_{1,it} \quad \sum_{t=1}^{T} x_{2,it} \quad \ldots \quad \sum_{t=1}^{T} x_{k,it}\right]' \tag{5}$$

Part (e)
Now consider the linear regression model $y_{it}|\mathbf{x}_{it}, \bar{y}_i = E(y_{it}|\mathbf{x}_{it}, \bar{y}_i) + \epsilon_{it}$. Show that the error term ϵ_{it} has a conditional mean of zero.

Part (f)
Using your results from parts (d) and (e), show that this regression model can be written in deviation-from-the-mean form. Estimating $\boldsymbol{\beta}$ using this model will be equivalent to which well-known panel data model? Can you provide some intuition?

March Macro QE question
Consider an endowment economy composed of infinitely lived individuals. Each individual has an endowment at each time t of $y_t = y$. Each individual pays a time t lump sum tax to the government equal to τ_t.

Government spending per person at time t is g_t.

There is an international bond market open to individuals and the government. The internation interest rate is fixed at $r = \theta$. Bonds are one period loans. A bond purchased at time t pays (1+r) units of output at t+1. b_t is an individual's *demand* for bonds and b_t^g is the government's *supply* of bonds.

The current time period is t=1. Individual consumption is c_t.

Budget Constraints:

Government: $b_t^g = g_t - \tau_t$, t=1 and $b_t^g = (1+r)b_{t-1}^g + g_t - \tau_t$, t>1.

Each individual: $b_t = y - c_t - \tau_t$, t=1 and $b_t = (1+r)b_{t-1} + y - \tau_t - c_t$.

The Utility Function

Each individual seeks to maximize $\sum_{t=1}^{\infty}\{[\ln(c_t)]/(1+\theta)^{t-1}\}$.

(a) Assume $g_1 = \tau_1$ and $g_t = \tau_t = 0$, t>1. Find an individual's optimal values for c_t and b_t as functions of y, $r = \theta$, and $g_1 = \tau_1$. Explain the intuition for your results.

In (a) assume the government cuts τ_1 to $\tilde{\tau}_1$ but leaves g_1 unchanged, so that $g_1 > \tilde{\tau}_1$. Of course this requires the future taxes must increase above zero. For t>1, assume the government increases τ_t from 0 to $\tilde{\tau}$>0, an amount sufficient to finance the time 1 tax cut.

(b) Find the value for $\tilde{\tau}$.

(c) Describe the time path for b_t^g.

(d) Find the effects on the optimal values for c_t and b_t, $t \geq 1$. Explain the intuition for your answers.

Costly tax collection

The per person, real cost of collecting τ_i taxes from an individual is $C(\tau_i)$, with $C'>0$ and $C''>0$. ***Net*** time i taxes are $\hat{\tau}_i=\tau_i-C(\tau_i)$. ***Net*** taxes are available to finance government.

Assume g_1 is constant and $g_t=0$, t>1. The present value of total tax collection costs is

$$\sum_{i=1}^{\infty} C(\tau_i)/(1+r)^{i-1}$$

It must be the case that

$$\sum_{i=1}^{\infty} (\hat{\tau}_i)/(1+r)^{i-1} = g_1.$$

(e) The optimal time path for taxes, τ_i, will maximize the lifetime utility of each individual. Determine and characterize the optimal tax vector. Explain the intuition behind your results.

(f) Determine the optimal time path of government debt, b_i^g. Explain the intuition behind your answer.

(g) What does this example say about the effects of a time 1 tax cut?

Spring 2015 Qualifying Exam Question

Simple RBC Model with Capital

The goal of the social planner is to maximize the lifetime utility of the representative household

$$U_t = E_t \sum_{j=0}^{\infty} \beta^j \ln C_{t+j} \quad , 0 < \beta < 1$$

subject to the constraints that

$$K_{t+1} = Y_t - C_t,$$

$$Y_t = A_t K_t^{\alpha}, \qquad 0 < \alpha < 1, \qquad K_0 > 0.$$

The first constraint is just the market equilibrium condition. The second constraint is the production function, and some initial amount of capital K_0 is given. The log of the productivity shock is just white noise $\ln A_t = \varepsilon_t$, where ε_t has mean zero and some finite variance.

a) Set up the Lagrangian and show that the Euler equation for this problem is

$$\beta E_t \left(\frac{C_t}{C_{t+1}} \alpha A_{t+1} K_{t+1}^{\alpha-1} \right) = 1.$$

b) Consider the solution $C_t = \gamma Y_t$, where γ is an unknown constant. Using the Euler equation in a), solve γ in terms of the parameters of the model α and β.

c) Does consumption have a larger or smaller variance than output?

d) Show that log output follows an AR(1) process.

e) Write down the steady state value of log output. Suppose log output is at its steady state value up until period t when there is a one-time production shock $\varepsilon_t = 1$ (and $\varepsilon_{t+1} = \varepsilon_{t+2} = \cdots = 0$). Plot the time path of log output from period t onwards.

f) If α is closer to one, how is your graph in e) affected?