**Department of Economics and the Department of Agricultural and Applied Economics**

**Ph.D. Qualifying Exam, August 2018**

**Part 1: Microeconomics**

**3 Questions, 2 pages**

**Note: The minutes assigned to each question indicate the weight given to the question. For example, question 1 is 50 minutes out of a total of 180 minutes and thus counts for 5/18 of the grade for this exam.**

# PROPOSAL MICRO QE, AUGUST 2018

Problem 1 (50 minutes)
Suppose that $\succeq$ on $X = \mathbb{R}^2_{++}$ is represented by the following utility function:

$$u(x_1, x_2) = 2 \cdot (x_1)^{\frac{1}{2}} + \frac{4}{3} \cdot (x_2)^{\frac{3}{4}}.$$

Assume that $p_1, p_2, m > 0$.

- Show that $|MRS|$ is equal to the price ratio at the optimal consumption bundle.
- Determine the Walrasian demand correspondence $x(\mathbf{p}, m)$.
- Check if $x(\mathbf{p}, m)$ is homogeneous of degree zero.
- Check if Walras' law is satisfied.
- Check if goods are normal or inferior.
- Check if goods are substitutes or complements.
- Check if the (uncompensated) Law of Demand is satisfied.

Problem 2 (60 minutes)
An auctioneer wants to sell an object by a first-price, sealed-bid auction. There are $n = 2$ bidders, where each bidder $i \in \{1, 2\}$ has a private valuation $\theta_i$ independently drawn from a binary distribution. In particular, a) $\theta_1 \in \{2, 8\}$ and $Pr(\theta_1 = 2) = \frac{1}{2}$ and b) $\theta_2 \in \{4, 6\}$ and $Pr(\theta_2 = 4) = \frac{1}{2}$. Each player $i$ submits $b_i \in \{1, 2, \ldots, 7, 8\}$, and the bidder with the highest bid wins the object and pays his/her bid (if there are more than one highest bidder, then the winner is randomly determined by the auctioneer).

- Describe this game as a Bayesian game.
- Write down the expected payoff of player $i$ for given $\theta_i$ and strategy profile $(s_i, s_j)$.
- Write down the conditions for BNE of this game.
- Find all pure strategy BNE.[†]

---

[†]For simplicity, one can assume that $b_1(8) \geq b_2(6) \geq b_2(4) \geq b_1(2)$ where $b_i(\theta)$ is player $i$'s bid when his/her private valuation is $\theta_i$.

Problem 3 (70 minutes)

Consider a pure exchange economy $\mathcal{E} = \{(X^i, \succsim^i, \omega^i)_{i=1}^I\}$ with two consumers $i \in \{1, 2\}$ and two commodities $l \in \{1, 2\}$. Each consumer $i \in \{1, 2\}$ chooses among commodity bundles in $X^i = \mathbb{R}^2_+$ according to his/her preferences $\succsim^i$ described by the following utility function:

$$u(x_1^i, x_2^i) = \min\{2x_1^i + x_2^i, x_1^i + 2x_2^i\}.$$

The initial endowment of consumer 1 is given by $\omega^1 = (6, 0)$ and that of 2 by $\omega^2 = (0, 12)$. Let $p_1$ be the price of good 1 and $p_2 = 1$ be the price of good 2.

1. For consumer 1, draw some indifference curves in a diagram.
2. Determine the marginal rate of substitution of consumer 1.
3. Determine the demand functions for consumer 1, i.e., $x_1^1(p_1, p_2)$ and $x_2^1(p_1, p_2)$.
4. Determine the demand functions for consumer 2, i.e., $x_1^2(p_1, p_2)$ and $x_2^2(p_1, p_2)$.
5. Draw the Edgeworth box for this economy.
6. In the Edgworth box, illustrate the set of Pareto efficient allocations.
7. Determine the Walrasian market equilibrium for this economy, i.e, the equilibrium allocation $\tilde{x}$ and the equilibrium price vector $p^* = (p_1^*, p_2^*)$.

**Department of Economics and the Department of Agricultural and Applied Economics**

**Ph.D. Qualifying Exam, August 2018**

**Part 2: Econometrics and Macroeconomics**

**Part 2A: Econometrics (3 Questions on 3 pages, 20 minutes each)**
**Part 2B: Macroeconomics (1 hour, 2 Questions on 2 pages)**

**Note: Econometrics counts for 60% of this exam (with each question being weighted equally) and macro for 40%.**

**Macroeconomics (1 hour)**

In the questions to follow, explain your answers fully.

[1] Individuals are identical and each individual maximizes $\sum_{t=1}^{\infty} \ln(c_t)/(1+\theta)^{t-1}$ and each has an endowment of output at time $t$ equal to $y$. Time t government purchases are $g_t$ and are financed with lump sum taxes $\tau_t = g_t$.

Utility is maximized subject to the constraint $\sum_{t=1}^{T} c_t \Pi_t = \sum_{t=1}^{T} \hat{y}_t \Pi_t$.

$\Pi_1 = 1$, $\Pi_2 = (1+r_1)^{-1}$; $\Pi_t \equiv \{(1+r_1)\times.....\times(1+r_{t-1})\}^{-1}$, $t>2$; $\hat{y}_t$ is net after-tax income.

Suppose initially $g_t = g$, a constant.

(a) Net output is $\hat{y}_t = y - \tau_t$. Suppose there is an increase in $g_1$. What is the effect on the time 1 equilibrium interest rate?

(b) Suppose there is an increase in $g_t$ for each $t \geq 1$. What is the effect on the time 1 equilibrium interest rate?

Assume government spending increases output (think "infrastructure," highways, bridges and the like). As noted $g_t$ is financed with time $t$ lump sum taxes, $\tau_t = g_t$. The increase in time $t$ output is $2(g_t)^{1/2}$. Time $t$ output is $y_t + 2(g_t)^{1/2}$ and total net output is $ny_t$ (output minus taxes) (Assume $y \geq g_t$.)

$$\hat{y}_t = y - g_t + 2(g_t)^{1/2}.$$

(c) What is the optimal level of $g_t$? That is, what level of $g_t$ will maximize time t *net* output? Call the optimum $g^*$.

(d) Suppose at time 1 $g_1 < g^*$. Find the effect of a marginal increase in $g_1$ on the equilibrium time 1 interest rate.

(e) Suppose at time 1 $g_1 > g^*$. Find the effect of a marginal increase in $g_1$ on the equilibrium time 1 interest rate.

[2] Time t output is $y_t = B(k_t)^{\alpha}(g_t)^{\beta}$, in which equation, $g_t$ government provided "infrastructure," $\alpha+\beta<1$. Infrastructure has a depreciation rate of 1, so at each time t there is no carry over of infrastructure from the past.

(a) Time $t$ net output is $ny_t = y_t - g_t$. What value of $g_t$ maximizes net output? Call the solution $g_t$*.

(b) Consider a Solow model in which $\partial k_t / \partial t = s(y_t) - \delta k_t$. Find the steady state value of $k_t$.

(c) Suppose $\alpha+\beta=1$. Find the growth rate of $k_t$. What is the growth rate of $y_t$?

Question **1** (suggested time: **20** minutes).

**(a)** In the context of the simple (one parameter) Normal model (table 1), define an optimal $\alpha$-significance level test $T_\alpha$ for the hypotheses:

$$H_0\colon \mu \leq \mu_0, \text{ vs. } H_1\colon \mu > \mu_0. \tag{1}$$

**(b)** Define and explain the following properties of a Neyman-Pearson (N-P) test: (i) Uniformly Most Powerful, (ii) Unbiased, (iii) Consistent.

**(c)** Define the large $n$ problem and explain how it affects the p-value and the N-P accept/reject rules in light of your answer in (a)..

**(d)** State the fallacies of acceptance and rejection and relate your answer to the sample size $n$.

| **Table 1 - The simple** (one parameter) **Normal model** | | |
|---|---|---|
| Statistical GM: | $X_t = \mu + u_t,\ t\in\mathbb{N}$, | |
| [1] | Normal: | $X_t \backsim \mathsf{N}(.,.)$, |
| | i.e. $f(x;\theta)=\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$, $\mu\in\mathbb{R}$, $x\in\mathbb{R}$ | |
| [2] | Constant mean: | $E(X_t)=\mu$, for all $t\in\mathbb{N}$, |
| [3] | Constant variance: | $Var(X_t)=\sigma^2$ (known), |
| [4] | Independence: | $\{X_t,\ t\in\mathbb{N}\}$ - independent process. |

# Question 2 (suggested time: 20 minutes)

Consider the CLRM $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ (call it "Model 1"). Let $\mathbf{X}$ be partitioned into $\mathbf{X}_1$ and $\mathbf{X}_2$, which have dimensions $n$ by $k_1$ and $n$ by $k_2$, respectively. Let $k = k_1 + k_2$. Partition $\boldsymbol{\beta}$ accordingly into $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$.

Assume $\mathbf{X}_1$ and $\mathbf{X}_2$ are perfectly orthogonal in the sample and in the population. Furthermore, neither of them are correlated with the regression error, by the usual CLRM assumption.

**Part (a), 8 points**

Using partitioned regression results, derive separate estimators for $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ (call them $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$).

For the full model, express the residual vector $\mathbf{e}$ as a function of $\mathbf{y}$ and the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, where $\mathbf{P}_j = \mathbf{X}_j \left(\mathbf{X}_j'\mathbf{X}_j\right)^{-1} \mathbf{X}_j' \quad j = 1, 2$.

Show that $(\mathbf{P}_1 + \mathbf{P}_2)$ is idempotent under the model assumptions.

Express the sum of squared residuals (SSR) as a function of $\mathbf{y}$ and the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$. Call it $SSR_1$.

**Part (b), 8 points**

Now consider a second CLRM model ("Model 2") that regresses $\mathbf{y}$ only on $\mathbf{X}_1$, i.e. $\mathbf{y} = \mathbf{X}_1\boldsymbol{\gamma} + \boldsymbol{\nu}$. Write down the OLS solution for $\hat{\boldsymbol{\gamma}}$ and compare it to your estimator for $\boldsymbol{\beta}_1$ from part (a). Comment.

For Model 2, express the residual vector $\mathbf{e}$, as well as the sum of squared residuals (SSR), as a function of $\mathbf{y}$ and the projection matrix $\mathbf{P}_1$. Call this sum $SSR_2$.

Show the expression for the difference of the two SSRs, and argue that the SSR from Model 1 can be no larger than the SSR from Model 2. (*Hint: Recall that projection matrices are semipositive definite*).

**Part (c), 4 points**

What does this imply for the estimate of the (conditional) variance of $\hat{\boldsymbol{\beta}}_1$ compared to the (conditional) variance of $\hat{\boldsymbol{\gamma}}$ for both a finite sample of size $n$, and when $n \to \infty$?

*(Hint: Take a close look at the expression for the estimated error variance, $s^2$, for each model.)*

# Question 3 (suggested time: 20 minutes)

You are involved in a research project on VT students' commute to campus. Specifically, you are interested in the share of students that commute by (i) walking/biking (WB), (ii) bus (B), or (iii) car (C). You survey a sample of $n$ randomly chosen students, and and ask them how they got to campus that day (let's assume that their transportation choice for the interview day is their typical way to commute). You find that $y_1$ walked or biked, $y_2$ used the bus, and $y_3$ got to campus by car, with $y_j > 0, \forall j$, and $\sum_{j=1}^{3} y_j = n$.

Your aim is to use this sample and Bayesian analysis to estimate the population shares for these three commuting types for all VT students, labeled $\pi_1$, $\pi_2$, and $\pi_3$, with $0 \leq \pi_j \leq 1$, and $\sum_{j=1}^{3} \pi_j = 1$.

You start by specifying a *multinomial* likelihood for the sample, given as:

$$p(\mathbf{y}|\boldsymbol{\pi}) = \left(\frac{n}{\prod_{j=1}^{3} y_j!}\right) \prod_{j=1}^{3} \pi_j^{y_j} \quad \text{with}$$

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}',$$

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}', \quad \sum_{j=1}^{3} y_j = n, \quad \text{and} \quad \sum_{j=1}^{3} \pi_j = 1,$$

As a prior for $\boldsymbol{\pi}$ you choose a Dirichlet distribution. The density and expectation for the Dirichlet are given as:

$$p(\boldsymbol{\pi}) = \left(\frac{\Gamma\left(\sum_{j=1}^{3} \alpha_j\right)}{\prod_{j=1}^{3} \Gamma(\alpha_j)}\right) \prod_{j=1}^{3} \pi_j^{\alpha_j - 1}, \quad \text{where}$$

$$\alpha_j > 0, \ \forall j, \quad \text{and}$$

$$E(\pi_j) = \frac{\alpha_j}{\sum_{j=1}^{3} \alpha_j}$$

**Part (a)**
Derive the *kernel* of the posterior distribution $p(\boldsymbol{\pi}|\mathbf{y})$, and determine the statistical distribution for the full posterior. Show the posterior parameters for this distribution.

**Part (b)**
Assume you have commuting information from other "college towns" like Blacksburg, with average proportions of 0.1, 0.4, and 0.5 for WB, B, and C, respectively. Interpreting these averages as prior expectations, and letting $\sum_{j=1}^{3} \alpha_j = 10$, derive the prior parameters $\alpha_1$, $\alpha_2$, and $\alpha_3$.

**Part (c)**
Assume your VT sample of 100 students produces $y_1 = 17$, $y_2 = 52$, and $y_3 = 31$. Using all the information from above, compute the posterior expectations for the population shares.

The town of Blacksburg is willing to sponsor new walking / biking trails if the expected population proportion of walkers/bikers exceeds 15%. What will be the town's decision?