Problem 1 (40 minutes)

Let $X = \mathbb{R}^2_+$, $\Upsilon = \mathbb{R}^2_{++} \times \mathbb{R}_{++}$ be the set of strictly positive price-income pairs $(\mathbf{p}, m)$ and $\mathbb{B}$ be the budget correspondence, that is $\mathbb{B}(\mathbf{p}, m) = \{\mathbf{x} \in X \,|\, \mathbf{px} \le m\}$ for $(\mathbf{p}, m) \in \Upsilon$. Suppose that $\succsim$ is a rational, locally nonsatiated, continuous, and strictly convex preference relation on X. Given $\succsim$, let $\mathbf{x}(\mathbf{p}, m)$ be the Walrasian demand correspondence; i.e.,

$$\mathbf{x}(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{B}(\mathbf{p}, m) \,|\, \mathbf{x} \succsim \mathbf{y} \text{ for any } \mathbf{y} \in \mathbb{B}(\mathbf{p}, m)\}.$$

a) Show that $\mathbf{x}(\mathbf{p}, m)$ satisfies Walras' law; i.e., $p_1\, x_1(\mathbf{p}, m) + p_2\, x_2(\mathbf{p}, m) = m$.

b) Show that $\mathbf{x}(\mathbf{p}, m)$ is a singleton.

c) Show that for any $(\mathbf{p}, m)$ and $(\mathbf{p}', m')$:

$$\text{If } \mathbf{x}(\mathbf{p}, m) \in \mathbb{B}(\mathbf{p}', m') \text{ and } \mathbf{x}(\mathbf{p}, m) \ne \mathbf{x}(\mathbf{p}', m'), \text{ then } \mathbf{x}(\mathbf{p}', m') \notin \mathbb{B}(\mathbf{p}, m).$$

Problem 2 (40 minutes)

Consider a pure exchange economy $\mathcal{E} = \{(X^i, \succsim^i, \omega^i)_{i=1}^I\}$ with two consumers $i \in \{1, 2\}$, two commodities $l \in \{1, 2\}$ and the initial endowments: $\omega^1 = (0.5, 3)$ and $\omega^2 = (3.5, 1)$.

Consumer i chooses among commodity bundles in $X^i = \mathbb{R}^2_+$ according to preferences $\succsim^i$ represented by the following utility function u^i:

$$u^i(x_1^i, x_2^i) = \begin{cases} 2x_1^i + x_2^i - 2 & \text{if } x_1^i \le 1; \\ \dfrac{1}{2}x_1^i + x_2^i - \frac{1}{2} & \text{if } x_1^i > 1. \end{cases}$$

Let $p \in \mathbb{R}^2_+$, $p \ne 0$, denote a price vector.

(A) Draw in a diagram the two indifference curves given by $u^i(x_1^i, x_2^i) = 1$ and $u^i(x_1^i, x_2^i) = 2$.

(B) Indicate in an Edgeworth Box the set of Pareto optimal allocations.

(C) Walrasian Equilibria of $\mathcal{E}$:

(C.1) Which of the following price pairs $p = (p_1, p_2)$ constitute an equilibrium price system?
$p^I = (3, 1)$;
$p^{II} = (2, 1)$;
$p^{III} = (1, 1)$;
$p^{IV} = (1, 2)$;
$p^V = (1, 3)$.

(C.2) Describe formally the set of equilibrium allocations.

(C.3) Depict in an Edgeworth Box the initial endowment, the equilibrium budget line(s) and the equilibrium allocations.

Problem 3 (50 minutes)
Let Y be a production technology characterized by the following production function:

$$f(z) = \begin{cases} n & \text{if } z \in \mathbb{R}_+ \text{ and } n^2 \leq z < (n+1)^2, \text{ with } n \in \{0, 1, \ldots, 5\}; \\ 5 & \text{if } z > 25. \end{cases}$$

Let $p \in \mathbb{R}_{++}$ be the output price and $w \in \mathbb{R}_{++}$ the input price.

1. Draw the corresponding technology $Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 \leq f(-y_1)\}$ in a diagram.
2. Determine which of the following properties are satisfied by the technology Y:
   - (i) closedness
   - (ii) no free lunch,
   - (iii) possibility of inaction,
   - (iv) free disposal,
   - (v) irreversibility,
   - (vi) nonincreasing returns to scale,
   - (vii) nondecreasing returns,
   - (viii) constant returns to scale,
   - (ix) additivity,
   - (x) convexity.
3. Derive the profit function $\pi(p, w)$ and the supply function (or correspondence) $y_2(p, w)$.
4. How would $\pi(p, w)$ and $y_2(p, w)$ change if f was replaced by

$$\widehat{f}(z) = n \text{ if } z \in \mathbb{R}_+ \text{ and } n^2 \leq z < (n+1)^2, \text{ with } n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}?$$

Problem 4 (50 minutes)

Every hour, $N = 6000$ car drivers who go from A to D. A car driver wants to get from A to D as fast as possible. Travel time is measured in minutes. A driver's utility is given by $u(t) = -t$ if it takes the driver t minutes to go from A to D. Therefore, the driver's objective is to minimize t.

(a)

Originally, a driver has the choice between two routes, ABD and ACD, as depicted in Figure 1, with the following travel times:

- It takes $t_{AB} = \frac{1}{100} \cdot x$ minutes to go from A to B if x cars per hour use road AB.
- It takes $t_{BD} = 50 + \frac{1}{1000} \cdot x$ minutes to go from B to D if x cars per hour use road BD.
- It takes $t_{AC} = 50 + \frac{1}{1000} \cdot x$ minutes to go from A to C if x cars per hour use road AC.
- It takes $t_{CD} = \frac{1}{100} \cdot x$ minutes to go from C to D if x cars per hour use road CD.

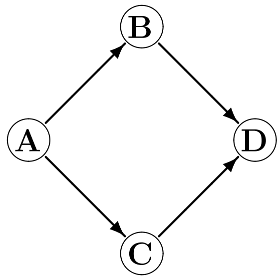

Figure 1

Let x_{ABD} be the number of drivers who choose route ABD each hour and x_{ACD} the number of drivers who choose route ACD each hour.

☐ Determine x_{ABD} and x_{ACD} in Nash equilibrium (in pure strategies).

☐ Determine a driver's total travel time in a Nash equilibrium (in pure strategies).

(b)

Suppose that a new one-way road from B to C opens as depicted in Figure 2, so that now a driver can choose between three routes to go from A to D: ABD, ACD and ABCD.

- It takes $t_{BC} = 10 + \frac{1}{1000} \cdot x$ minutes to go from B to C if x cars per hour use road BC.

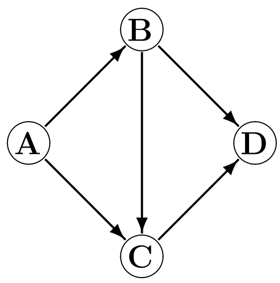

Figure 2

Let x_{ABCD} be the number of drivers who choose route ABCD each hour.

☐ Show that there exists a Nash equilibrium with $x_{ABD} = x_{ACD} = x_{ABCD} = 2000$.

☐ Determine a driver's total travel time in such an equilibrium.

(c)

Suppose that the one-way road from B to C is replaced by a two-way road as in Figure 3, so that travel both from B to C and from C to B is possible. Then a driver can choose between four routes to go from A to D: ABD, ACD, ABCD, and ACBD.

• Like before, it takes $t_{BC} = 10 + \frac{1}{1000} \cdot x$ minutes to go from B to C if x cars per hour go from B to C.

• It takes $t_{CB} = \alpha + \beta \cdot x$ minutes, with $\alpha \geq 0$ and $\beta \geq 0$, to go from C to B if x cars per hour go from C to B.

Let x_{ACBD} be the number of drivers who choose route ACBD each hour.

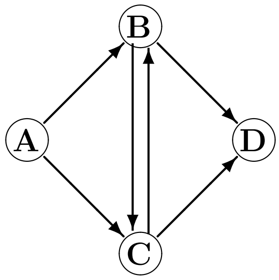

Figure 3

☐ Show that there exists a Nash equilibrium with
$x_{ABD} = x_{ACD} = x_{ABCD} = 2000$, $x_{ACBD} = 0$.

Econometrics Qualifying Exam, Nov. 2020

Question 1 (20 minutes)

(a) Consider the simple Normal model:

$$X_t \backsim \mathsf{NIID}(\mu, \sigma^2), \ \ t{=}1, 2, ..., n, ...,$$

with σ^2 is <u>known</u>; 'NIID' stands for 'Normal, Independent and Identically Distributed'.

(i) Explain why the following sampling distribution for the standardized $\overline{X}_n{=}\frac{1}{n}\sum_{t=1}^{n} X_t$:

$$\tfrac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \backsim \mathsf{N}(0, 1), \tag{1}$$

can be interpreted in two different ways depending on the nature of the reasoning applied.

(ii) Using your answer in (i) explain how (1) provides the basis for testing the hypotheses:

$$H_0\colon \ \mu \leq \mu_0, \text{ vs. } \quad H_1\colon \ \mu > \mu_0, \tag{2}$$

to give rise to an optimal Neyman-Pearson (N-P) test based on the test statistic $d(\mathbf{X}){=}\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\sigma}$, and what 'optimal' means in this case.

(b) (i) State the sampling distributions of $d(\mathbf{X})$ under both the null and alternative hypotheses.

(ii) Using your answer in (b(i) define: [i] type I error probability, [ii] power of the test and [iii] the p-value, and [iv] compare and contrast (i) and (iii).

(c) Using your answer in (a) explain how (1) can be used to construct a $(1{-}\alpha)$ Confidence interval for μ^*, the 'true' value of μ, whatever that happens to be.

(d) State the fallacies of acceptance and rejection and explain why the accept/reject rules and the p-value are vulnerable to these fallacies when they are interpreted as providing evidence for or against the null or the alternative.

Question 2 (suggested time: 20 minutes)

Assume you collect samples of equal size n for each of $g = 1 \dots G$ groups. For each group you observe an n by 1 outcome variable $\mathbf{y}_g$, and k explanatory variables collected in n by k matrix $\mathbf{X}_g$.

Assume that in truth, outcomes for each group follow a *separate CLRM* with separate coefficient vectors $\boldsymbol{\beta}_g$, typical error properties, and *common variance*, that is:

$$\begin{aligned} \mathbf{y}_g &= \mathbf{X}_g\boldsymbol{\beta}_g + \boldsymbol{\epsilon}_g, \quad g = 1 \dots G \\ \boldsymbol{\epsilon}_g &\sim n\left(\mathbf{0}, \sigma^2\mathbf{I}_n\right) \end{aligned} \tag{1}$$

Part (a)

1. Write down the OLS estimator for the g^{th} model - call it generically $\mathbf{b}_g$.
2. Show that it is unbiased for its corresponding true parameter vector $\boldsymbol{\beta}_g$.

Part (b)

1. Consider an *equivalent* single regression model that stacks the G vectors of dependent variables into $\tilde{\mathbf{y}} = \left[\mathbf{y}_1' \quad \dots \mathbf{y}_G'\right]'$:
$$\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\epsilon}}. \tag{2}$$
Show the explicit form of $\tilde{\mathbf{X}}$, $\tilde{\boldsymbol{\beta}}$, and $\tilde{\boldsymbol{\epsilon}}$ in terms of the original model components in equ. (1). Recall that the model has to allow for separate coefficient vectors for the two groups.
2. Show that the OLS solution for this model, call it $\tilde{\mathbf{b}}$, is equivalent to that obtained from estimating the G group-specific models separately.

Part (c)

Now consider a (wrong) model that assumes *equal coefficient vectors* for all groups. Let the model be written as:

$$\tilde{\mathbf{y}} = \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\nu}, \tag{3}$$

where, as before, $\tilde{\mathbf{y}} = \left[\mathbf{y}_1' \quad \dots \quad \mathbf{y}_G'\right]'$, and $\boldsymbol{\gamma}$ is k by 1.

1. Show the explicit form for $\mathbf{X}$.
2. Given the form of the *correct* single model derived in the previous part, show the explicit contents of the error vector $\boldsymbol{\nu}$.
3. Show that this error vector is correlated with $\mathbf{X}$, i.e. that $E\left(\mathbf{X}'\boldsymbol{\nu}\right) \neq \mathbf{0}$.
4. What does this imply for the OLS estimate of $\boldsymbol{\gamma}$ (call it $\hat{\gamma}$)?
5. Under what condition would the correlation between $\mathbf{X}$ and $\boldsymbol{\nu}$ go to zero?

Question 3 (suggested time: 20 minutes)

Consider the true CLRM, given at the observation level as:

$$\begin{aligned} y_i &= \alpha + \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i, \quad \text{with} \\ \epsilon_i &\sim n\left(0, \sigma_\epsilon^2\right) \quad \forall i, \end{aligned} \tag{1}$$

where $\mathbf{x}_i$ is a vector of explanatory variables, and the individual error term ϵ_i are, as usual, uncorrelated across observations and with any of the $\mathbf{x}_i$'s.

Assume that the dependent variable y_i is measured with error ν_i, such that

$$\begin{aligned} \tilde{y}_i &= y_i + \nu_i, \quad \text{with} \\ \nu_i &\sim n\left(0, \sigma_\nu^2\right) \quad \forall i, \end{aligned} \tag{2}$$

where $\tilde{y}_i$ is the actually *observed* dependent variable used in the analysis, and n denotes the normal density. Further assume that ν_i is uncorrelated with $\mathbf{x}_i$ and ϵ_i for all observations.

Part (a)

(a) Write the true model in (1) at the sample level (you can call the data matrix $\mathbf{X}$). Then write the observational model, which uses $\tilde{\mathbf{y}}$ instead of $\mathbf{y}$, at the full-sample level. You can use $\mathbf{i}$ to denote a vector of 1's, and $\mathbf{I}$ to denote the identity matrix.

(b) At the sample level, show the *true form* of the error vector for the observational model, as well as its expectation and variance. Call this error vector $\boldsymbol{\omega}$.

(c) Using partitioned regression, show that this model, despite the mis-measured dependent variable, produces unbiased estimates for both the intercept α and the coefficient vector $\boldsymbol{\beta}$.

(d) In summary, what is the only disadvantage for estimation introduced by this type of measurement problem?

Part (b)

Now assume, instead, that the measurement error in (2) takes the following form:

$$\nu_i \sim n\left(\mu, \sigma_\nu^2\right), \quad \mu \neq 0 \tag{3}$$

(a) Show the expectation and variance for $\boldsymbol{\omega}$ in this case.

(b) Show that this model produces an unbiased estimate for the coefficient vector $\boldsymbol{\beta}$, but a biased estimate for the intercept.

(c) Building on your results from above, show how this unbiasedness property of $\boldsymbol{\beta}$ would change if, instead, we had $\nu_i \sim n\left(\mu_i, \sigma_\nu^2\right)$. You can use the notation of $\boldsymbol{\mu} = \begin{bmatrix}\mu_1 & \mu_2 & \dots & \mu_n\end{bmatrix}'$.