Problem 1 (45 minutes)
Olivia has the preference relation $\succsim$ on $\mathbb{R}^2_+$ representable by

$$u(x_1, x_2) = \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}.$$

(a) Sketch one of Olivia's indifference curves.

(b) Write the following definitions for a preference relation on $\mathbb{R}^2_+$:

- Convex,
- Strictly Convex,
- Homothetic.

(c) Which of these three properties do Olivia's preferences satisfy? In each case, provide justification in the form of a proof or a counterexample.

(d) Given a price vector $\mathbf{p} = (p_1, p_2)$ and income I, derive Olivia's Walrasian demand correspondence $x(\mathbf{p}, I)$.

Problem 2 (45 minutes)
Consider a firm with production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f(z) = \begin{cases} z & \text{for} \quad 0 \le z < 1; \\ 1 + \sqrt{z-1} & \text{for} \quad z \ge 1. \end{cases}$$

(a) Does the firm's technology have constant, decreasing or increasing returns to scale? Explain!

(b) Let $p = 1$ be the output price and $w > 0$ be the input price.
Solve the firm's profit maximization problem and give the firm's factor demand and maximal profit as functions of w, i.e., as $z(w)$ and $\pi(w)$, respectively.

Problem 3 (45 minutes)
Consider a quantity-setting duopoly with inverse market demand function

$$P(q) = \begin{cases} 1-q & \text{for } q \in [0,1]; \\ 0 & \text{for } q > 1 \end{cases}$$

where $q = q_1 + q_2$ is industry output.
The two firms play a two-stage game. At the first stage, firm 1 chooses its output $q_1 \geq 0$. It incurs costs $C_1(q_1) = q_1^2$.
At the second stage, firm 2 chooses its output $q_2 \geq 0$. It incurs costs $C_2(q_2) = q_2^2$.
Firm i's profit is

$$\Pi_i(q_1, q_2) = P(q_1 + q_2)q_i - q_i^2.$$

Now let $q_0 \in (0, 3/7)$ and consider the following pair of strategies:

$$q_1 = q_0; \; q_2(q_1) = \begin{cases} \frac{1}{4}(1-q_0) & \text{if } q_1 = q_0; \\ \\ 1 & \text{if } q_1 \neq q_0. \end{cases}$$

(a) Show that this pair of strategies constitutes a Nash equilibrium of the two-stage game for any choice of $q_0 \in (0, 3/7)$.

(b) Are any of these Nash equilibria subgame perfect? Explain!

(c) Determine the pairs of strategies that are subgame perfect equilibria.

Problem 4 (45 minutes)
Let $L \geq 1$. Consider a finite economy $\mathcal{E} = ((X_i, \omega_i, \succsim_i, \theta_i)_{i \in I}, (Y_j)_{j \in J})$ with L commodities $\ell = 1, \ldots, L$, $n \geq 1$ consumers $i \in I = \{1, \ldots, n\}$ and $m \geq 1$ producers (firms) $j \in J = \{1, \ldots, m\}$.
For $i \in I$,

- $X_i = \mathbb{R}^L_+$ is i's consumption set. A consumption bundle $x_i \in X_i$ assumes the form $x_i = (x_1^1, \ldots, x_2^n)$ where x_i^ℓ is i's consumption of commodity ℓ;
- $\omega_i = (\omega_i^1, \ldots, \omega_i^n) \in X_i$ is i's endowment bundle;
- $\succsim_i$ is i's preference relation, with standard properties;
- $\theta_i = (\theta_{i1}, \ldots, \theta_{im}) \geq 0$ is the vector of i's property shares in firms.

For $j \in J$,

- $Y_j \subseteq \mathbb{R}^L$ is j's technology (production set), with $0 \in Y_j$;
- $\sum_{i \in I} \theta_{ij} = 1$.

For a price system $p = (p_1, \ldots, p_\ell) \gg 0$ and $j \in J$, let

- $y_j(p)$ denote j's profit maximizers, that is the solutions of the problem $\max p y_j$ s.t. $y_j \in Y_j$;
- $\pi_j(p) \in \mathbb{R}_+ \cup \{+\infty\}$ denote j's maximum profit.

Next consider the aggregate technology

$\widehat{Y} = \sum_{j \in J} Y_j = \{\widehat{y} \in \mathbb{R}^L \mid \widehat{y} = \sum_{j \in J} y_j \text{ for some } y_1 \in Y_1, \ldots, y_m \in Y_m\}$.

For a price system $p = (p_1, \ldots, p_\ell) \gg 0$, let

- $\widehat{y}(p)$ the profit maximizers with respect to the aggregate technology, that is the solutions of the problem $\max p\widehat{y}$ s.t. $\widehat{y} \in \widehat{Y}$;
- $\widehat{\pi}(p)$ denote the maximum profit with respect to the aggregate technology.

You may use without proof the following **aggregation property**, known from class and the textbook:

- $\widehat{y}(p) = \sum_{j \in J} y_j(p)$, that is
  $y_1 \in y_1(p), \ldots, y_m \in y_m(p) \Longrightarrow \sum_{j \in J} y_j \in \widehat{y}(p)$ and
  $\widehat{y} \in \widehat{y}(p) \Longrightarrow \exists y_1 \in y_1(p), \ldots, y_m \in y_m(p)$ s.t. $\widehat{y} = \sum_{j \in J} y_j$;
- $\widehat{\pi}(p) = \sum_{j \in J} \pi_j(p)$.

From now on, let a competitive or Walrasian equilibrium $(p^*; x^*, y^*)$ of the economy $\mathcal{E}$ be given, with equilibrium price system $p^* \gg 0$.

Suppose the m firms are merged into one, with technology $\widehat{Y}$, and the consumers receive ownership shares $\widehat{\theta}_i \geq 0,\ i \in I$, in the merged firm, so that $\sum_{i \in I} \widehat{\theta}_i = 1$.

(a) Suppose $\sum_{j \in J} \pi_j(p^*) > 0$. Determine ownership shares $\widehat{\theta}_i,\ i \in I$, so that the economy $\widehat{\mathcal{E}} = ((X_i, \omega_i, \succsim_i, \widehat{\theta}_i)_{i \in I}, (\widehat{Y}_j)_{j \in J})$ has the equilibrium $(p^*; x^*, \widehat{y}^*)$ with $\widehat{y}^* = \sum_{j \in J} y_j^*$.
(This means that the merger does not matter for equilibrium consumption if the after-merger property rights are assigned correctly.)

Explain your answer!

HINT. The outcome is obtained if each consumer's income under the price system p^* is the same before and after the merger.

(b) What if $\sum_{j \in J} \pi_j(p^*) = 0$?

(c) Solve the problem in (a) numerically for the special case $L = 2$, $n = 2$, $m = 2$, $\theta_{11} = 1, \theta_{22} = 1$, and technologies

$$Y_1 = \left\{ (y_1^1, y_1^2) \in \mathbb{R}^2 \;\middle|\; y_1^1 \leq 0,\ y_1^2 \leq 2 \cdot \sqrt{-y_1^1} \right\},$$

$$Y_2 = \left\{ (y_2^1, y_2^2) \in \mathbb{R}^2 \;\middle|\; y_2^1 \leq 0,\ y_2^2 \leq \sqrt{2} \cdot \sqrt{-y_2^1} \right\}.$$

In this case, consumer 1 is the sole owner of firm 1 and consumer 2 is the sole owner of firm 2.

HINT. Determine and compare $\pi_1(p)$ and $\pi_2(p)$ for $p \gg 0$.

Question 1 (20 minutes)

Consider the simple Normal model:

$$X_t \backsim \mathsf{NIID}(\mu,\sigma^2),\ \ t{=}1,2,...,n,...,$$

with σ^2 is <u>known</u>; 'NIID' stands for 'Normal, Independent and Identically Distributed'.

(a) (i) Statistics textbooks often write the sampling distributions of the pivotal function $d(\mathbf{X};\mu)$ as follows:

$$d(\mathbf{X};\mu){=}\tfrac{\sqrt{n}(\overline{X}_n-\mu)}{\sigma} \backsim \mathsf{N}(0,1),\ \text{ where } \overline{X}_n{=}\tfrac{1}{n}\textstyle\sum_{t=1}^{n}X_t. \tag{1}$$

Explain how it is derived. In particular, explain why the mean in $\mathsf{N}(0,1)$ is zero using the underlying reasoning.

(ii) Explain how (1), when properly interpreted, can be used to construct an optimal $(1{-}\alpha)$ Confidence Interval for μ; explain what an 'optimal' CI refers to in this case.

(b) (i) When testing the hypotheses:

$$H_0\colon\ \mu \leq \mu_0,\ \text{vs.}\quad H_1\colon\ \mu > \mu_0, \tag{2}$$

explain how the following sampling distribution of the test statistic $d(\mathbf{X})$ is derived:

$$d(\mathbf{X}){=}\tfrac{\sqrt{n}(\overline{X}_n-\mu_0)}{\sigma} \backsim \mathsf{N}(0,1), \tag{3}$$

and how it is used, when properly interested to derive the type I error probability and the p-value.

(ii) Explain why the sampling distribution of $d(\mathbf{X})$ under $\mu_1 > \mu_0$ takes the form:

$$d(\mathbf{X}){=}\tfrac{\sqrt{n}\left(\overline{X}_n-\mu_0\right)}{\sigma} \overset{\mu=\mu_1}{\backsim} \mathsf{N}(\delta_1,1),\ \ \delta_1{=}\tfrac{\sqrt{n}(\mu_1-\mu_0)}{\sigma},\ \text{ for all } \mu_1{>}\mu_0, \tag{4}$$

and use your answer to define the type II error probability and the power of this test.

(iii) Explain what it takes to define a Neyman-Pearson (N-P) test, and construct an optimal test for the hypotheses (2). Explain what 'optimal' refers to in this case.

(c) State the fallacies of acceptance and rejection and explain why the accept/reject rules and the p-value are vulnerable to these fallacies when they are interpreted as providing evidence for or against the null or the alternative.

Question 2

Suggested time: 20 minutes

Consider the Pareto density, generically given as:

$$f(y) = \theta\alpha^{\theta}y^{-(1+\theta)}, \quad \text{with} \\ \alpha \leq y < \infty \\ \theta, \alpha > 0, \tag{1}$$

where α is a ***known*** location parameter, and θ is an unknown shape parameter. As captured in the second line of (1), α also determines the lower bound of the support of y.

Part (a)

1. Show formally that, for a given α and θ:
   - i $f(y)$ is maximized at $y = \alpha$
   - ii $f(y)$ is downward sloping w.r.t. y
   - iii $f(y)$ is convex to the origin w.r.t. y
2. Sketch this density graphically, with $f(y)$ on the y-axis, and y on the x-axis.

Part (b)

Consider a sample of n observations, y_i, $i = 1 \ldots n$, generated by this Pareto density.

1. Write the log-likelihood for the i^{th} observation and the full sample.
2. Derive the gradient for the sample and solve (analytically) for the MLE estimator of θ (call it $\hat{\theta}$).
3. Does the second order condition confirm that your estimator of θ does indeed maximize the likelihood? Explain.

Part (c)

The Pareto density is related to the exponential density as follows:

$$ln(y/\alpha) \sim Exp\left(\theta^{-1}\right) \tag{2}$$

That is, if y follows the Pareto as outlined above, the natural log of y, divided by location parameter α, follows the exponential density with parameter θ^{-1}. This further implies that:

$$E(ln(y/\alpha)) = \theta^{-1} \tag{3}$$

where $E(.)$ is the expectation operator.

1. Building on (3), and using very simple math (no integration required!!!) derive $E(ln(y))$.
2. Using this result, show that the *score identity* holds for the Pareto likelihood and it's gradient, derived in the previous part.

Question 3

Suggested time: 20 minutes

As starting point for this question, recall the law of iterated expectation, which states that for any two random variables, say z and v, we have:

$$E(z) = E_v(E(z|v)) \tag{1}$$

Similarly, by the decomposition of variance theorem we have:

$$V(z) = V_v(E(z|v)) + E_v(V(z|v)) \tag{2}$$

Now consider a Bayesian problem where θ is a parameter of interest, and y denotes (generically) the observed data. Assume the likelihood function is given by $p(y|\theta)$, and the prior is written as $p(\theta)$, as usual.

Part (a)

1. Using the framework from above, show in mathematical terms and describe in words the relationship between prior expectation, $E(\theta)$, and the posterior expectation $E(\theta|y)$.
2. Given this relationship, why or why not would you expect the posterior expectation to be similar to the prior expectation for a *single draw of the data*, say $\tilde{y}$?
3. Does this relationship tell us anything about the relative magnitude of prior and posterior expectation?

Part (b)

1. Now use the decomposition of variance theorem from above to relate prior and posterior variances in mathematical terms.
2. Assuming none of the variance terms involved are zero, does this relationship tell us anything about the relative magnitude of the prior variance and the *expected* posterior variance? Why or why not?
3. In that sense, would you prefer a small or large variation of the *posterior mean* over all manifestations of the data?

Part (c)

Similar relationships can also be found between prior expectation, posterior expectation, and sample moments for an empirical application. To illustrate, consider the binomial likelihood that describes the probability of obtaining y "success" draws out of a sample of n repetitions ("trials"):

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{(n-y)} \tag{3}$$

Assume a Beta prior for θ, that is:

$$\begin{aligned}
p(\theta) &= \frac{\Gamma(\alpha_0 + \beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} \theta^{(\alpha_0 - 1)} (1-\theta)^{(\beta_0 - 1)}, \quad \text{with} \\
&\alpha_0, \beta_0 > 0, \\
E(\theta) &= \frac{\alpha_0}{\alpha_0 + \beta_0}
\end{aligned} \tag{4}$$

where α_0 and β_0 are the prior shape parameters.

1. Show that the kernel of the posterior distribution $p(\theta|y, n)$ characterizes another Beta distribution. Show the form of the posterior shape parameters (call them α_1 and β_1) and the posterior expectation in terms of the prior parameters, y, and n.

2. Show that the posterior expectation of θ can be written as a weighted average between the prior expectation and the sample proportion of successes.

3. As the number of repetitions ("trials") gets larger, which component of this weighted average will start to dominate, and why?

Macroeconomics Qualifier Examination

August 2021

Time allocated: 70 minutes.

Question 1:
(35 minutes)

Consider an economy consisting of a constant population of infinitely lived individuals. A representative individual seeks to maximize lifetime expected utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where β represents the discount factor. The instantaneous utility function is given by $u(c_t) = c_t - \theta c_t^2$, $\theta > 0$. We assume that $u'(c) > 0$. An individual produces output using $y_t = Ak_t + e_t$, where A is a constant and k_t represents capital stock in possession of the individual. The term e_t represents a stochastic shock in the output production that follows a first-order autoregressive process: $e_t = \varphi e_{t-1} + \varepsilon_t$, where $-1 < \varphi < 1$ and where ε_ts are mean zero i.i.d shocks. At each period, an individual faces the budget constraint given by $y_t = c_t + (k_{t+1} - k_t)$.

(a) Write down the Bellman's functional equation.
(b) Solve an individual's optimization problem and lay out the steps in detail to arrive at the Euler's equation that relates c_t and c_{t+1}.
(c) Define $E_t[c_{t+1} - c_t]$ as the expected change in consumption between the periods $t+1$ and t (conditional on the available information set in period t). Derive $E_t[c_{t+1} - c_t]$ when $\beta(1 + A) = 1$.

Question 2:
(35 minutes):

Consider an economy in which output is produced each period using private and public capital according to the production function, $y = k_1^{\alpha} k_2^{1-\alpha}$, where $0 < \alpha < 1$. k_1 and k_2 denote private and public capital per capita. Both public and private capital are subjected to a given common depreciation rate δ. A representative household's utility function is given by $U_t = \int_0^{\infty} e^{-\rho t} \log c_t \, dt$. The capital accumulation equation for private capital is given by $\dot{k_1} = k_1^{\alpha} k_2^{1-\alpha} - \delta k_1 - (c + \tau)$, where c and τ represent private consumption and lump-sum tax paid by a household to the government. Assume that government balances its budget (i.e., $g = \tau$). Also assume that the entire tax revenue is used as an investment to public capital so that $\dot{k_2} = \tau - \delta k_2$.

a. Set up and solve the optimization problem from the point of view of a central planner who seeks to maximize the welfare of a representative household.
b. Show that the central planner will maintain a constant ratio of private to public capital stock along the optimal path.
c. What is the equilibrium growth rate for consumption in this centralized economy?