Problem 1 (40 minutes)
Let $Y \subset \mathbb{R}^3$ be a production set (technology) where commodities 1 and 2 serve as inputs to produce commodity 3. Y is given by the following production function:

$$f(z_1, z_2) = \begin{cases} 2\sqrt{\min\{z_1, z_2\}} - 2 & \text{if } z_1 \geq 1 \text{ and } z_2 \geq 1; \\ 0 & \text{otherwise .} \end{cases}$$

Let $p \in \mathbb{R}_{++}$ be the output price and $(w_1, w_2) \in \mathbb{R}^2_{++}$ the pair of input prices.

1. What returns to scale does the technology display?

2. Derive the profit function $\pi(p, w_1, w_2)$ and the supply correspondence $y(p, w_1, w_2)$.

Problem 2 (30 minutes)
Lotteries or probability distributions over outcomes in $S = \{1, 2, 3\}$ correspond to probability vectors in the simplex
$\Delta = \{p = (p(1), p(2), p(3)) \in \mathbb{R}^3_+ | \sum_{s=1}^3 p(s) = 1\}$.

Consider the following four elements of Δ,

$$p^1 = (0, 1/2, 1/2), p^2 = (1/8, 1/4, 5/8), p^3 = (1/3, 1/3, 1/3), p^4 = (11/24, 1/12, 11/24).$$

(a) Asterix has a preference relation $\succsim_a$ on Δ that satisfies

$$p^1 \sim_a p^4 \succ_a p^2 \sim_a p^3.$$

(b) Beatrix has a preference relation $\succsim_b$ on Δ that satisfies

$$p^1 \sim_b p^3 \succ_b p^2 \sim_b p^4.$$

Which of the two decision makers cannot be expected utility maximizer(s)? Explain!

NOTE that for $i = a, b$:
If the preference relation $\succsim_i$ on Δ is represented by an expected utility function $U_i(p) = u_i p$ with $u_i = (u_i(1), u_i(2), u_i(3))$, then the assumed indifferences impose linear restrictions on the components $u_i(s), s \in S$.

Problem 3 (50 minutes)

PART I:

The economy of the island of Atlantix is a pure exchange economy with $L = 2$ goods labeled $k = 1, 2$ and $N = 2n$ consumers labeled $i = 1, \ldots, N$. All consumers have the same consumption set $\mathbb{R}^2_+$ and the same Cobb-Douglas preferences represented by the utility function

$$u_i(x_i^1, x_i^2) = x_i^1 x_i^2$$

for any consumer i and consumption bundles $x_i = (x_i^1, x_i^2) \in \mathbb{R}^2_+$.

There are two types of consumers:

The n consumers $i = 1, \ldots, n$ have the endowment bundle
$\omega_i = (\omega_i^1, \omega_i^2) = (20, 20)$.

The n consumers $i = n + 1, \ldots, 2n$ have the endowment bundle
$\omega_i = (\omega_i^1, \omega_i^2) = (20, 0)$.

■ Determine the competitive equilibrium allocation $(x_i^*)_{i=1,\ldots,2n}$ for this economy.

HINT. You may use the fact that all consumers have identical homothetic preferences.

PART II:

The government wants to convert Atlantix into a wildlife reserve. It proposes to resettle the inhabitants of Atlantix to the hitherto uninhabited island Pacifix. On Pacifix, the settlers will enjoy the same two goods, will have the same preferences and each will have the endowment bundle $\widetilde{\omega}_i = (\widetilde{\omega}_i^1, \widetilde{\omega}_i^2) = (21, 21)$. Since each will have a larger endowment than before, the government proposal seems attractive.

Suppose all consumers have moved from Atlantix to Pacifix. Then some will find that, indeed, they are better off than before while others will be worse off and disappointed.

■ In order to prove this claim, do the following:

(a) Determine the competitive equilibrium allocation $(x_i^{**})_{i=1,\ldots,2n}$ for the new Pacifix economy.

(b) Compare the equilibrium utilities $u_i(x_i^*)$ and $u_i(x_i^{**})$ for $i = 1, \ldots, 2n$.

Problem 4 (60 minutes)
This problem concerns an endowment destruction game. Consider two consumers $i = 1, 2$ with the following characteristics.

Consumer 1	Consumer 2
consumption set $X_1 = \mathbb{R}^2_+$	consumption set $X_2 = \mathbb{R}^2_+$
initial endowment $\omega_1 = (8, 0)$	initial endowment $\omega_2 = (0, 2)$
Leontieff preferences represented by $v_1(x_1^1, x_1^2) = \min\{x_1^1, x_1^2\}$	Cobb-Douglas preferences represented by $v_2(x_2^1, x_2^2) = x_2^1 x_2^2$

A consumer has the option to destroy part of his initial endowment. After destruction, the two consumers have **reduced endowments**

$$e_1 = (a, 0) \text{ with } 1 < a \leq 8, \quad e_2 = (0, b) \text{ with } 0 < b \leq 2,$$

and enter the market with these endowments. The pure exchange economy

$$\mathcal{E}(a, b) \;=\; (X_i, v_i, e_i)_{i=1,2}$$

has a unique competitive equilibrium allocation, with equilibrium consumption $x_i(a, b)$ for consumer $i = 1, 2$.

(a) Determine the Nash equilibria in pure strategies of the endowment destruction game

$$\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$$

with $I = \{1, 2\}$, $S_1 = (1,\ 8]$, $S_2 = (0,\ 2]$, $u_i(a, b) = v_i(x_i(a, b))$ for $(a, b) \in S_1 \times S_2$, $i \in I$.

(b) What can be said about equilibrium payoffs?

Question **1** (suggested time: **20** minutes).

(a) State the Neyman-Pearson (N-P) lemma for an optimal test, and explain why invoking this lemma in the context of the simple Bernoulli model:

$$X_t \backsim \mathsf{BerIID}\left(\theta,\ \theta(1-\theta)\right),\ \theta\in(0,1),\ t\in\mathbb{N}:=(1,2,3,...,n,...)$$

to derive an optimal test for the simple hypotheses:

$$H_0\colon\ \theta{=}.5,\ \text{vs.}\quad H_1\colon\ \theta{=}\tfrac{18}{35}, \tag{1}$$

constitutes a misuse of the lemma.

(b) (i) Explain how the result of the N-P lemma extends to more realistic cases so long as the null and the alternative hypotheses constitute a partition of the parameter space.

(ii) Explain how you would reformulate the null and alternative hypotheses in (1) to accord with the archetypal N-P formulation with a view to avoid fallacious results.

(c) Explain the following properties of N-P tests: (i) Uniformly Most Powerful test, (ii) Unbiased test, and (iii) Consistent test.

(d) Explain why the frequentist error probabilities, including the type I, II, coverage and the p-value, CANNOT be conditional probabilities.

Question 2

(suggested time: 20 minutes)

Consider the following regression model:

$$\mathbf{y} = \beta_0 + \beta_1 \mathbf{x} + \boldsymbol{\epsilon} \tag{1}$$

where all vectors are of length n and the β's are, respectively, a scalar intercept and coefficient. The single explanatory variable is continuous, but potentially endogenous. You decide to use a single instrument, $\mathbf{z}$, to obtain an unbiased estimate of β_1.

The instrument $\mathbf{z}$ is binary - n_0 of its elements take the value of 0, and the remaining n_1 elements take the value of 1. For convenience, you can assume that the data are sorted such that the zero elements are listed first.

Further define $\bar{y}$ as the mean of $\mathbf{y}$, $\bar{y}_0$ as the mean of $\mathbf{y}$ for the observations for which $z_i = 0$ (that is, the first n_0 observations), and $\bar{y}_1$ as the mean of $\mathbf{y}$ for the observations for which $z_i = 1$. Similarly define $\bar{x}$, $\bar{x}_0$, and $\bar{x}_1$.

Part (a)

Show that:

1. $n\bar{z} = n_1$
2. $\bar{y} = \frac{n_0\bar{y}_0 + n_1\bar{y}_1}{n}$
3. $\sum_i^n z_i y_i = n_1 \bar{y}_1$
4. $\sum_i^n z_i x_i = n_1 \bar{x}_1$

Part (b)

Write down the generic matrix form of the simple IV estimator for this model (call it $\boldsymbol{\beta}_{IV}$), using $\mathbf{Z} = \begin{bmatrix}\mathbf{i} & \mathbf{z}\end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix}\mathbf{i} & \mathbf{x}\end{bmatrix}$.

Letting $\boldsymbol{\beta}_{IV} = \begin{bmatrix}\hat{\beta}_0 & \hat{\beta}_1\end{bmatrix}'$, show that $\hat{\beta}_1$ can be written as $\hat{\beta}_1 = \frac{\sum_i^n z_i y_i - n\bar{z}\bar{y}}{\sum_i^n z_i x_i - n\bar{z}\bar{x}}$.

Hint: recall that the inverse of a 2 by 2 matrix $\mathbf{A} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is given as $\frac{1}{|\mathbf{A}|}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$.

Part (c)

Using your results from part (a), show that this can be further simplified to $\hat{\beta}_1 = \frac{\bar{y}_1 - \bar{y}_0}{\bar{x}_1 - \bar{x}_0}$.

Part (d)

Can you provide some intuition for this result?

To make things more concrete, lets assume that y denotes wage, x denotes days spent in a training program, and z denotes if a sibling has also taken the training program.

Question 3
(suggested time: 20 minutes)

Assume you collect samples of equal size n for each of two groups, female (F) and male (M). For each group you observe an n by 1 outcome variable $\mathbf{y}_j$, $j = F, M$, and k explanatory variables collected in n by k matrix $\mathbf{X}_j$, $j = F, M$.

Assume that in truth, outcomes for each group follow a *separate CLRM* with the typical error properties and *common variance*, that is:

$$\begin{aligned}
&\mathbf{y}_F = \mathbf{X}_F \boldsymbol{\beta}_F + \boldsymbol{\epsilon}_F \\
&\mathbf{y}_M = \mathbf{X}_M \boldsymbol{\beta}_M + \boldsymbol{\epsilon}_M \\
&\boldsymbol{\epsilon}_F \sim n\left(\mathbf{0}, \sigma^2 \mathbf{I}_n\right), \quad \boldsymbol{\epsilon}_M \sim n\left(\mathbf{0}, \sigma^2 \mathbf{I}_n\right) \\
&\boldsymbol{\beta}_F \neq \boldsymbol{\beta}_M
\end{aligned} \tag{1}$$

Part (a)

1. Write down the OLS estimator for each model - call them $\mathbf{b}_F$ and $\mathbf{b}_M$, respectively.
2. Show that they are unbiased for their corresponding true parameter vectors.

Part (b)

1. Consider an *equivalent* single regression model that stacks the two vectors of dependent variables into $\tilde{\mathbf{y}} = \left[\mathbf{y}_F' \quad \mathbf{y}_M'\right]'$:
$$\tilde{\mathbf{y}} = \tilde{\mathbf{X}} \tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\epsilon}}. \tag{2}$$
Show the explicit form of $\tilde{\mathbf{X}}$, $\tilde{\boldsymbol{\beta}}$, and $\tilde{\boldsymbol{\epsilon}}$ in terms of the original model components in equ. (1). Recall that the model has to allow for separate coefficient vectors for the two groups.
2. Show that the OLS solution for this model, call it $\tilde{\mathbf{b}}$, is equivalent to that obtained from estimating the two group-specific models separately.

Part (c)
Now consider a (wrong) model that assumes equal coefficient effects on both groups. Let the model be written as:
$$\tilde{\mathbf{y}} = \mathbf{X} \boldsymbol{\gamma} + \boldsymbol{\nu}, \tag{3}$$
where, as before, $\tilde{\mathbf{y}} = \left[\mathbf{y}_F' \quad \mathbf{y}_M'\right]'$, and $\boldsymbol{\gamma}$ is k by 1.

1. Show the explicit form for $\mathbf{X}$.
2. Given the form of the *correct* single model derived in the previous part, show the explicit contents of the error vector $\boldsymbol{\nu}$.
3. Show that this error vector is correlated with $\mathbf{X}$, i.e. that $E\left(\mathbf{X}'\boldsymbol{\nu}\right) \neq \mathbf{0}$.
4. What does this imply for the OLS estimate of $\boldsymbol{\gamma}$ (call it $\hat{\boldsymbol{\gamma}}$)?
5. Under what condition would the correlation between $\mathbf{X}$ and $\boldsymbol{\nu}$ go to zero?

Macroeconomics Qualifier Examination

June 2021

Time allocated: 2 hours

Consider an economy where households have logarithmic utility function:

$$U = \int_0^\infty e^{-\rho t}\, lnC(t)dt$$

Each household has one unit of labor endowment supplied inelastically to the market for ruing wage rate, $w(t)$. There is L number of households in the economy. A household's budget constraint is given by $\dot{a}(t) = r(t)a(t) + w(t) - c(t)$, where $r(t)$, $a(t)$, and $c(t)$ denote time t interest rate, asset amounts, and consumption, respectively.

(a) Set up the household's optimization problem and find the optimal growth rate of consumption.

Assume that the output in this economy is produced according to

$$Y = L_y^{1-\alpha} \int_0^N x_j^\alpha dj$$

Where x_j represents intermediate good of variety j. The variable L_y represents the part of the total labor supply (L) that is used in the final good production. The rest of the labor supply, $(L - L_y)$ are used to innovate new varieties of intermediate goods. Let N(t) denotes the number of varieties of intermediate good available at time t. The number of varieties evolve according to

$$\frac{\dot{N}}{N} = \delta(L - L_y)$$

The profit of a (competitive) final good sector firm is given by

$$\pi = L_y^{1-\alpha} \int_0^N x_j^\alpha dj - wL_y - \int_0^N p_j\, x_j dj\ ,$$

where p_j and x_j represent the price and the quantity of the intermediate good of variety j, respectively.

Each intermediate good producer operates in a monopolistic setting where final good is the only input used in the production of intermediate goods. We assume that one unit of existing (one for which a blueprint is available) intermediate good requires one unit of final good to produce. We normalize the market price of final good to 1. Suppose that the government introduces a subsidy (τ_s) to the use of variable inputs in the production of intermediate goods. This is financed by a lump sum tax (and there is no need for you to be concerned about the government budget). As a result, intermediate good firms will only have to pay $\frac{1}{1+\tau_s}$ for each unit of final goods purchased to produce intermediate goods.

(b) Set up the final good producers' optimization problem and derive the demands for intermediate goods and the demand for labor.

(c) Find the profit maximizing price (p_j) and quantity (x_j) for a producer of intermediate good indexed (j).

(d) What value of τ_s the government could choose to eliminate the inefficiency arising due to the monopoly power of intermediate good producers?

Assume that labor can move freely between the final goods sector and the R&D sector. As a result, the costs of innovation of a new intermediate good is $\frac{w_t}{\delta N}$. Also, due to the free entry condition in the R&D sector, assume that the value of innovation of a new intermediate good is equal to the costs of innovation.

(e) Use the free entry (no-arbitrage) condition to establish that $r = \alpha\delta L_y$.

Finally, assume at that $\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N}$ holds for this economy.

(f) Use the above condition to find the expression for L_y in equilibrium.

(g) Show how the growth rate of this economy depends on the government subsidy, τ_s.