

Math Review Appendix

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I. Math Phobia? Mechanics versus Application

Do you have a math phobia? You are not alone. You may be wondering, “Why can’t we study this subject without the math?” We can, but we will not get very far in our understanding if we cower to the math. Try thinking of it this way. Math is a tool, like an iPhone, a blender, a guitar, or a circular saw. You first must learn how to operate the tool or device before you can do stuff with it; enjoy all its features. Of course, the fun is in applying the tool, not in learning how the tool works. It is not much fun just knowing how an iPhone works if you can’t use it to do interesting stuff. You can’t enjoy the music trapped inside a guitar if you do not know how to play it. The same is true with math. You first must learn how to use the tool, the mechanics, before you can have fun applying it. *So don’t confuse the mechanics of the tool with the application of the tool.* In the text we apply the tool. In this appendix we learn the mechanics of the tool. This appendix is like the operator’s manual for the math that will be used in this class. As with any operator’s manual, the best way to feel comfortable with the tool is to grab a cup of your favorite drink, get in a quiet place and read the manual very carefully, patiently, and practice. I promise if you will do this your math phobia will subside rather quickly. I am confident of this because the math we will be using is just middle school math; math you have probably just forgotten. You just need to review, but you must read carefully and be patient. With that, let’s get going.

II. It’s All About Relationships

Consider the following everyday type statements: “If I go to college, I’ll get a job,” “If I brush my teeth, I won’t have to wear dentures when I am older,” “if you don’t change the radio

station I will scream,” “If I cross the street when the light is red then I won’t get run over,” “If I learn to play the guitar all the girls will like me,” “If Virginia Tech makes the field goal they win the game,” “If I cut my hair this way then I’ll be more attractive and he may ask me out,” and “if I exercise daily I will be more healthy.” Do you notice a pattern? All of these statements have the same form or structure: “*if P happens then Q will happen.*” This type of reasoning or argument structure is formally known as either a **conditional statement** or a hypothetical syllogism. Conditional statements are ubiquitous in everyday life and are the foundational argument structure upon which all of science is based.

In science, the conditional statement is usually expressed as a relationship between variables. For example as discussed in chapter 1 of the book: “*if we increase the grams of carbohydrates by one gram then the calories will increase by 4 grams.*” Grams of carbohydrates and grams of calories are the two variables that are related. Or from nutrition/medicine: “*If the amount of LDL cholesterol is decreased then the probability of suffering a heart attack will decline.*” The two related variables here are the amount of LDL cholesterol and the probability of suffering a heart attack. Or from economics: “*If the price of gasoline increases then the quantity of gasoline sold will decrease.*” The price and quantity of gasoline are the two variables related. These are just a few examples and it would **not** be an oversimplification to say that this entire class is about the relationships between variables. This leads to the natural question: *How can we express or show the relationship between variables?*

There are generally three ways to demonstrate the relationship between variables:

1. Numerically – List in a table form different values of variables.
2. Graphically – Plot or graph the relationship between variables.

3. Mathematically – Use function or equation notation to state the relationship between variables.

Each of these methods has advantages and disadvantages, depending on the issue at hand and the purpose of showing the relationship between variables. However, due to the genius of some guy named René Descartes, it is now known that these are just different ways of expressing the same relationship; they are just different languages for expressing the relationship between variables. So if there is a mathematical form for a relationship between variables, it can be shown to have a graphical and numerical representation as well. In general there are two types of relationships: linear and nonlinear. These can further be partitioned into relationships between two variables and the relationship between more than two variables. We cover each of these cases in what follows.

III. Linear Relationships

In mathematics the number of variables is equivalent to the number of dimensions in the analysis. Thus if there are 2 variables there are then 2 dimensions. If there are more than two variables then there are more than 2 dimensions.

Linear Relationships with Two Variables

Let's start with an example you are already familiar with. As we discussed in the first chapter, every gram of protein translates into 4 grams of calories, *ceteris paribus*.¹ We can write this in its **mathematical form** (function) as $C_p = 4 \times N_p$. Now suppose we have 5 different

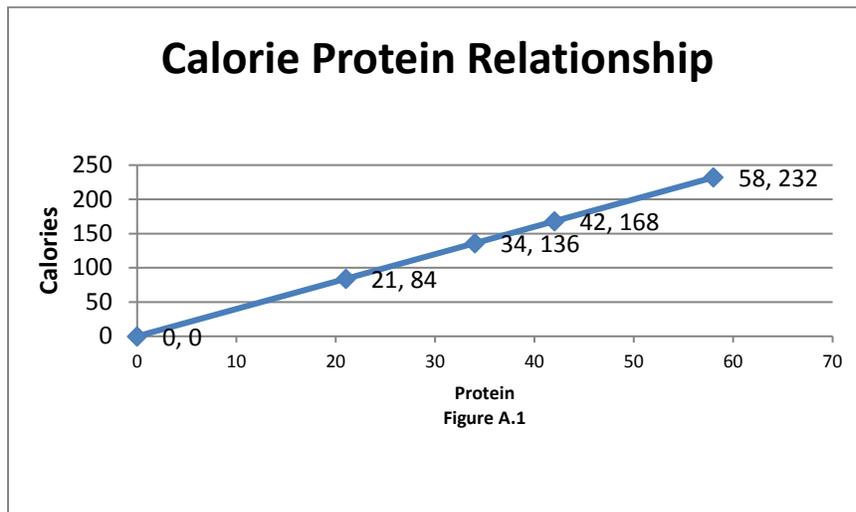
¹ The term *ceteris paribus* is Latin and means all else constant. See the methodology appendix in the book, specifically page 236, for more discussion.

protein levels: $N_p = 0, 21, 34, 42, 58$. If we substitute these into the equation we get the **numerical form** of this relationship or the table

Table A.1 Calories From Protein

N_p	C_p
0	0
21	84
34	136
42	168
58	232

Note we could alternatively present this relationship **graphically**. Because there are only two variables then we can use a two-dimensional graph as in figure A.1.



Let's put on our coal miner's hat and try to shed some light on these different ways of presenting these relationships considering each in isolation.

- Assume you just have the mathematical form. For the **mathematical form** one has to do some calculations to get the actual numbers and the general relationship (e.g., positive or negative) may not be immediately obvious to some. However, if one knows the mathematical form the other representations can be easily generated.

- ⦿ Assume you just have the numerical form. For the **numerical form** no calculations are necessary to determine the actual calorie value associated with a particular protein value, you just read off the table. However, the general relationship is not immediately obvious from just looking at the table and so one is not exactly sure what the value of calories would be for other values of protein (e.g., suppose protein level was 18 or 49).
- ⦿ Assume you just have the graphical form. For the **graphical form** the general relationship between the variables is visual and immediately obvious. It is positive: as protein increases, calories increases. However, like the numerical form, it is not clear what the exact value of the calories would be even if you knew the protein value, for some protein value *not* shown in figure (e.g., 50). It is also not clear if the observed relationship holds for other values of protein.

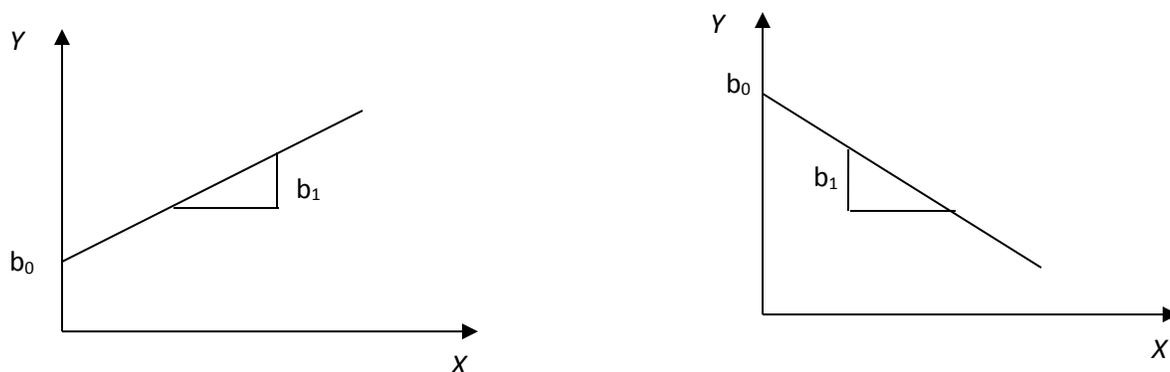
So each method (language) has some advantages and disadvantages.

This example is a special case of the more general relationship known as a linear equation with two variables. That is, recall from middle school mathematics that the formula for a line is $Y = b_0 + b_1 \times X$, which we write more simply as

$$(A.1) \quad Y = b_0 + b_1 X$$

Recall further that Y is called the **dependent variable** (a.k.a. left hand side variable, response variable, or an endogenous variable) and X is called the **independent variable** (a.k.a. a right hand side variable, explanatory variable, control variable, or an exogenous variable). The term b_0 is called the **intercept coefficient** and is the value of the dependent variable Y when the independent variable X is zero. The term b_1 is called the **slope coefficient** and indicates how much the dependent variable Y will change if the independent variable X is changed by one unit. Alternatively stated, the slope coefficient is the one unit “effect” of X on Y . Consequently, the

slope coefficient shows the relationship between X and Y . The coefficients (intercept and slope) are considered to be constants that do not change – they are not variables. Alternatively, X and Y are variables. If b_1 is positive then there is a positive relationship between X and Y . If b_1 is negative then there is a negative relationship between X and Y .



Panel A. Positive relationship $b_0 > 0, b_1 > 0$. Panel B. Negative relationship $b_0 > 0, b_1 < 0$.

Because the intercept and slope coefficient can be either negative, zero, or positive we need some way to indicate the “sign” (i.e., negative, zero, or positive) within the equation. We will do this by indicating under each coefficient if the sign is positive (+), zero (0), or negative (–). So for the general graph in panel A we would write the corresponding general equation as

$$(A.2) \quad Y = b_0 + b_1 X$$

(+ +)

NOTE: This (+) sign under the variable X does not mean X is positive or that it is increasing. It simply means there is a positive relationship between Y and X .

Think break!!! What would these two panels look like if $b_0 < 0$ and how would you write the equation?

How does this general formula relate to our protein-calorie example? Well in the protein-calorie example we had the equation $C_P = 4 \times N_P$. Note this is just then a special case of our general equation (A.2) with a little relabeling: Y to be C_p (or more simply $Y \equiv C_p$, where \equiv is just a quick way of saying ‘by definition’) and $X \equiv N_p$. This would give us $C_p = b_0 + b_1 N_p$, but this is still not our equation. However, if we also let the intercept in this equation be zero ($b_0 = 0$) and the slope be four ($b_1 = 4$), then we get our earlier equation $C_P = 4 N_P$ as a special case of the general equation. Note graphically the calorie-protein relationship shown earlier is just a special case of panel A. Of course numerically we just substitute the actual values in for $X \equiv N_p$ and we get the values for $Y \equiv C_p$.

Well these three different representations of the relationship between two variables are rather useful don’t you think? But there is an obvious limitation. In many cases more than one variable determines the value of some other variable. For example, we know that calories are actually determined by the amount of protein *and* the amount of carbohydrates *and* the amount of fat. In this case there are then four variables that are related (i.e., calories, protein, carbohydrates, and fat). Thus, we must extend the above analysis to beyond two variables or equivalently beyond two dimensions.

Linear Relationships with More Than Two Variables

Let’s again start with an example. In the previous 2 dimensional (variable) example we were implicitly assuming the level of carbohydrates and fats were each zero. This is obviously unrealistic because as we know from chapter 1 (equation 1.1), the **mathematical form** or formula for calculating calories is actually

$$Cals = 9 \times N_F + 4 \times N_C + 4 \times N_P.$$

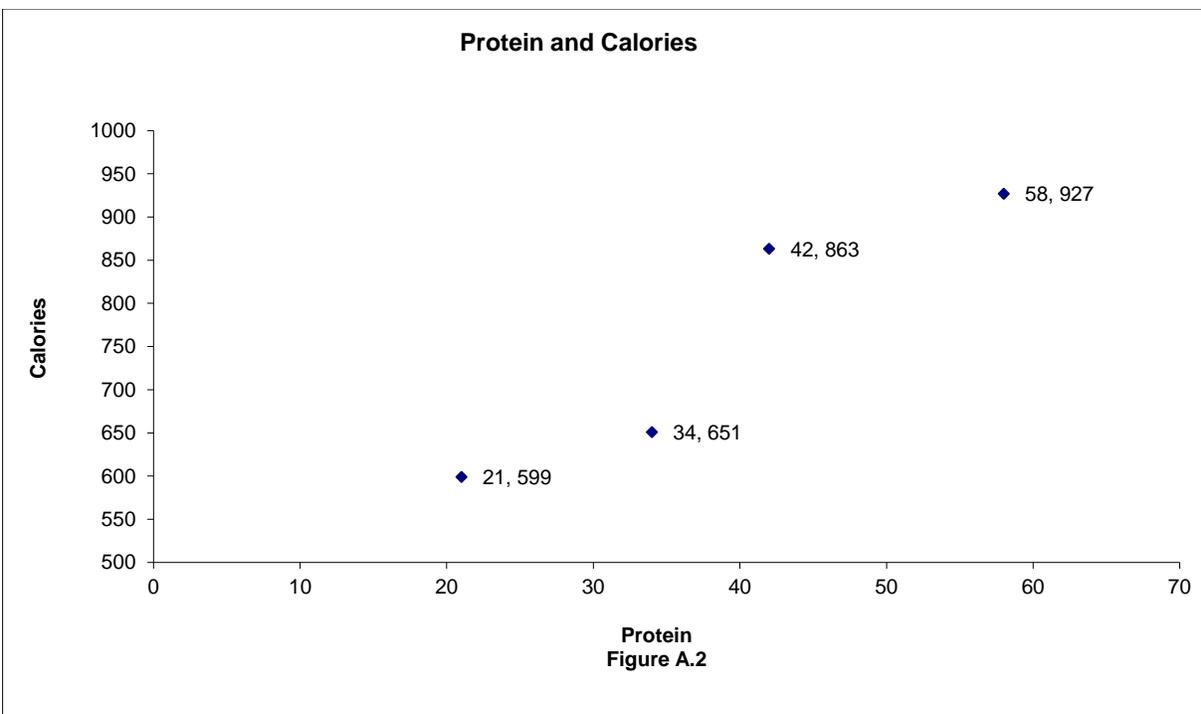
Let's repeat the exercise we did above. Suppose we had the following data for four different food items for their combinations for protein, carbohydrates, and fat. In **numerical form** we would have the following table. Note the (non zero) protein values are exactly the same as they were in table A.1.

Table A.2. Calories From Macronutrients			
<i>N_F</i>	<i>N_C</i>	<i>N_P</i>	<i>Cals</i>
35	50	21	599
35	50	34	651
55	50	42	863
55	50	58	927

However, clearly the calories are very different from table A.1. Furthermore, note for this table the main difference is with respect to protein, but importantly now the level of carbohydrates is not assumed zero but is constant at 50 across these items and more importantly there are two fat levels (35 and 55).

How would you display the information in table A.2 **graphically**? Remember, **A PIECE OF PAPER IS A TWO DIMENSIONAL SURFACE**. One dimension is up and down and the other dimension is left and right. Because it is a two dimensional surface and variables correspond to dimensions, then you can only graph two of these four variables at a time.² Let's first just plot protein against calories as we did in figure A.1 but now with this new data as shown in figure A.2.

² It is true you could draw a three dimensional surface on the two dimensional piece of paper, but this technically employs an optical illusion as there are really only two dimensions. The visual physical world as we know it consists of three dimensions: length, height, and depth. Mathematics can handle much higher dimensions (i.e. relationship between more than 3 variables) but they are difficult (impossible) to represent visually once we get beyond three dimensions.



Compare figure A.2 to figure A.1. What looks different? Yeah, the calorie values are different even though the protein values are the same. In figure A.1 when the points were connected it provided a straight line. If you connect the points in figure A.2 you don't get a straight line. Why? Simply stated, because the *ceteris paribus* condition is violated. All else is **not** the same or constant. The numbers in figure A.1 (and table A.1) are based on the assumption that the levels of carbohydrates and fat are zero. Obviously from table A.2 and figure A.2 they are not zero.

Let's see if we can figure out what is going on. Note if you go back to table A.2 there is no difference across these four items in terms of carbohydrates, they are all 50, so that can't be the difference. However, note that there are two levels of fat: 35 g and 55 g. Is this the culprit? Let's write out the calculations using the equation for each point – going left to right – in the graph.

First point:

$$\begin{aligned} 599 &= 9 \times 35 + 4 \times 50 + 4 \times 21 \\ &= 515 + 4 \times 21 \end{aligned}$$

Fourth point:

$$\begin{aligned} 927 &= 9 \times 55 + 4 \times 50 + 4 \times 42 \\ &= 695 + 4 \times 42 \end{aligned}$$

Second point:

$$\begin{aligned} 651 &= 9 \times 35 + 4 \times 50 + 4 \times 34 \\ &= 515 + 4 \times 34 \end{aligned}$$

Third point:

$$\begin{aligned} 863 &= 9 \times 55 + 4 \times 50 + 4 \times 58 \\ &= 695 + 4 \times 58 \end{aligned}$$

Look at these four cases carefully. Note both of the first two cases have the value 515 followed by the contribution of protein (i.e., $4 \times N_p$). However, the third and fourth cases have the value 695 followed by the contribution of protein (i.e., $4 \times N_p$). Thus if we wanted to we could write two equations for these four points.

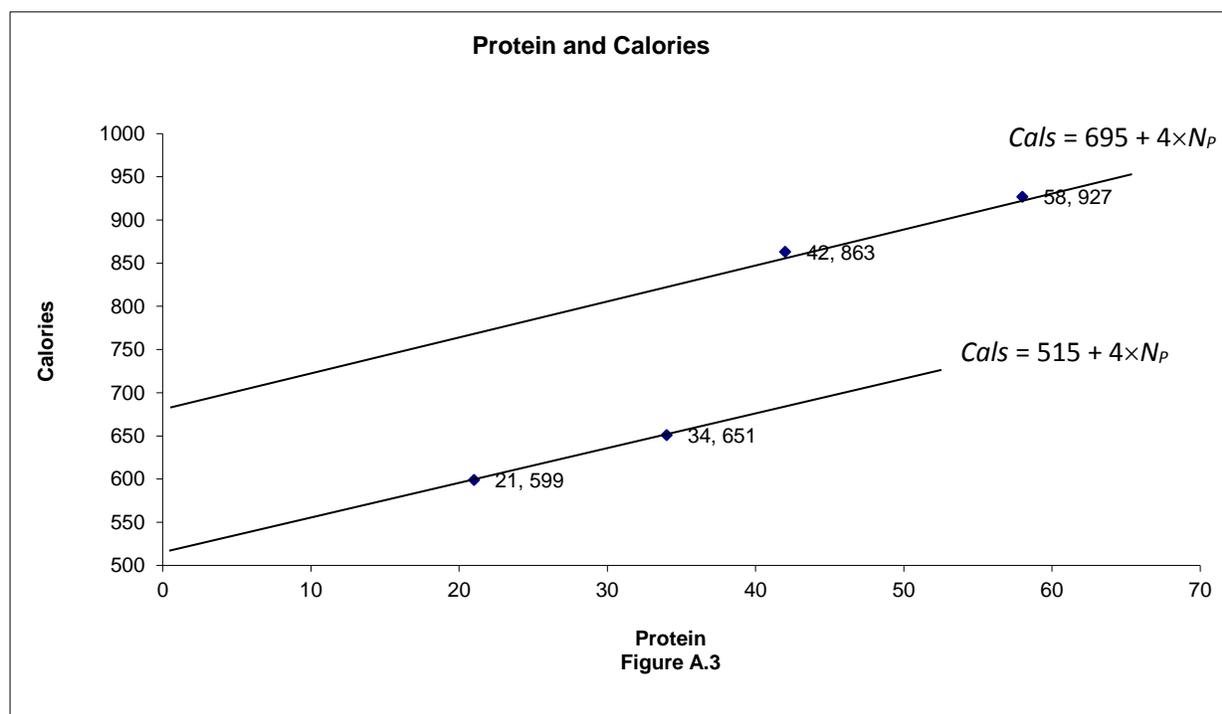
First and Second Point Equation:

$$Cals = 515 + 4 \times N_p$$

Third and Fourth Point Equation:

$$Cals = 695 + 4 \times N_p .$$

Hopefully you are saying, ‘Wait a minute Mr. Math Man’, these two equations only differ by the intercept: they have the same slope but just different intercepts. Using the general formula for a line $Y = b_0 + b_1 X$, in the first and second points the intercept is $b_0 = 515$ and the slope is $b_1 = 4$, whereas in the third and fourth points the intercept is $b_0 = 695$ and the slope is $b_1 = 4$. Let’s therefore plot these two equations.



Notice then these two equations are parallel. That should come as no surprise because the definition of **parallel lines** is lines that have the same slope but different intercepts. Why do they have different intercepts? If you go back and look at the data you will see that the difference is that the level of fat increased from the lower line to the upper line. Stated alternatively, the equation shifted up when the level of fat increased.

So what is the more general point to be taken from this example? (If you are asleep, wake up...**THIS IS EXTREMELY IMPORTANT**). When you draw a graph on a two dimensional surface, such as a piece of paper, you can only represent the relationship between two variables at a time. Thus movements along the curve show the relationship between the two variables shown on the vertical and horizontal axis. Changes in other variables must be represented by shifts in that curve.

Ok, now that you have seen a specific example let's talk about the more general case. Recall when we had two variables we wrote the equation for a line as $Y = b_0 + b_1 X$. Well now we are

going to extend this to many variables such that the **mathematical form** is

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + \dots + b_MX_M.$$

In this case there are M independent or explanatory variables and they are just numbered from 1 to M (i.e., $X_1, X_2, X_3, \dots, X_M$). The variable Y is still the dependent variable and b_0 is still the intercept coefficient and is the value of Y if **all** the explanatory variables are zero. Each of the other b s is again called a slope coefficient and represents the contribution a one unit increase in the corresponding X will have on Y or the one unit “effect” of the specific X on Y , *ceteris paribus* (i.e., holding all other variables constant).

Ok, are you starting to freak out? Don't. Take a deep breath. Chill out. You have already seen this at work because the equation $Cals = 9 \times N_F + 4 \times N_C + 4 \times N_P$ is just a special case of this general equation. How you say? Well just re-label things. Note if you substitute $Cals \equiv Y$, $N_F \equiv X_1$, $N_C \equiv X_2$, $N_P \equiv X_3$, $b_0 = 0$, $b_1 = 9$, $b_2 = 4$, $b_3 = 4$ into the general equation above you get the formula for calculating calories. Note in this example all of the values of the b s were positive, implying that if any of the X s increase the Y will increase, but this does not have to be the case. For example, suppose we are interested in the relationship between weight (Y), energy intake (X_1), and energy expenditure (X_2). In this case energy intake will increase weight but energy expenditure will *decrease* weight. In this case the value of the energy intake slope coefficient is positive ($b_1 > 0$) but the value of the energy expenditure slope coefficient is negative ($b_2 < 0$). To denote these different “effects” we will indicate under each coefficient if it is positive or negative such as follows

$$(A.3) \quad Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + \dots + b_MX_M.$$

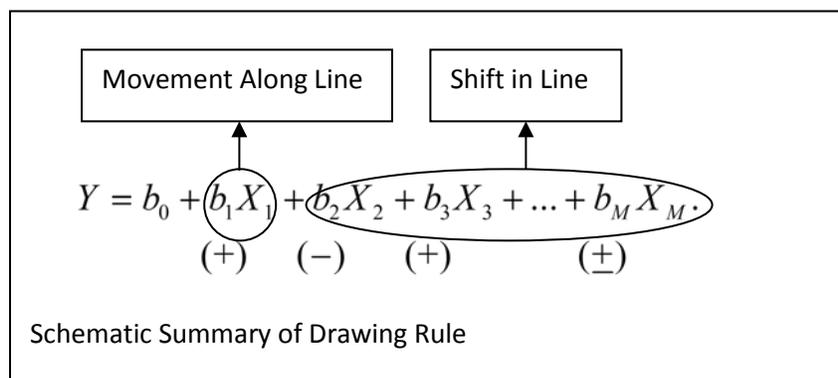
(+ (-) (+) (±)

In this case, the effect of X_1 on Y is positive ($b_1 > 0$), the effect of X_2 on Y is negative ($b_2 < 0$), the effect of X_3 on Y is positive ($b_3 > 0$), and the effect of X_M on Y is unknown– it could be positive or negative.

What about the **graphical** form of this equation, how do we show that? Proceed in the following stepwise fashion:

1. Choose the X variable you want to show related to Y (e.g., X_1)
2. Label the horizontal axis X and the vertical axis Y .
3. Look at the sign under the slope coefficient associated with the X variable you chose. If it is positive then draw a straight line from southwest to northeast. If it is negative draw a straight line from northwest to southeast. This graph shows the relationship between the X you chose and Y .
4. To show the effect on this graph of changing some other X variable, look at the sign under the slope coefficient of this other X variable. If it is positive, then increasing this variable will shift the line from step 3 out (to the right). If it is negative, then increasing this variable will shift the line from step 3 in (to the left).

A quick way to summarize these steps is to give the schematic



NOTE: Just by relabeling the variables the first explanatory variable (X_1) can be any variable and the actual signs under the variables will depend on the relationship being discussed.

Linear Isoquants

Often we will be interested in answering questions such as,

If I want to hold the level of calories constant how much would I have to decrease the amount of one food relative to another?

We implicitly ran into this question in chapter 7. Suppose we have two foods, with quantities in a common unit F_1 and F_2 . The common unit could be serving size, count, pieces, or grams.

Food one has c_1 calories per one unit and food two has c_2 calories per one unit. Total calories from the two foods can then be written as

$$Cals = c_1 F_1 + c_2 F_2.$$

Note we could solve this equation for the quantity of food one

$$F_1 = \frac{Cals}{c_1} - \frac{c_2}{c_1} F_2.$$

This equation is useful for telling us the tradeoff between food one and food two holding the quantity of calories constant. Specifically, the ratio of $c_2 \div c_1$ tells us the quantity of food one (F_1) you must give up for every additional unit of food two (F_2) you consume to hold caloric intake constant. For example, if a chicken breast from KFC has 320 calories but a drumstick has 160 calories, then for every chicken breast you eat, you must eat 2 less drumsticks (i.e., $2 = 320 \div 160$) to hold caloric intake constant. Thus the ratio $c_2 \div c_1$ measures the opportunity cost (what you give up) of consuming one more unit of F_2 in terms of *calories* of F_1 . Because we are holding the quantity calories constant this type of equation is often called an isoquant. The term “iso” is Greek for “equal,” so **isoquant** means “equal quantity”. In this case, holding the quantity of calories equal.

How do you get the isoquant in general? Simple. The same as you did in this example – just solve the equation like (A.3) for the variable you want. For example suppose you want the equation for X_1 . From middle school algebra just proceed in two steps:

1. Subtract $b_0 + b_2X_2 + b_3X_3 + \dots + b_MX_M$ from both sides of the equation (A.3) to yield

$$Y - b_0 - b_2X_2 - b_3X_3 - \dots - b_MX_M = b_1X_1.$$

(+ (-) (+) (±) (+)

2. Now divide both sides of this equation by b_1 to yield

$$(A.4) \quad X_1 = \frac{1}{b_1}Y - \frac{b_0}{b_1} - \frac{b_2}{b_1}X_2 - \frac{b_3}{b_1}X_3 - \dots - \frac{b_M}{b_1}X_M.$$

(+) (-) (-) (+) (±)

VERY IMPORTANT!!!: The sign under the coefficient in step 2 includes the minus sign and therefore you have to keep track of the signs. For example, note because b_2 is negative ($b_2 < 0$) so when you subtract b_2X_2 from both sides in step 1, the effect on b_1X_1 is positive (i.e., the negative of a negative is a positive). In the second step when you divide through by b_1 then if b_1 is positive ($b_1 > 0$) the “signs” do not change from step 1. However, if b_1 is negative ($b_1 < 0$) the “signs” do change from step 1.

Given all of this is extremely important that you understand, let’s pause and do an example.

Suppose you have the general linear equation notation for three variables

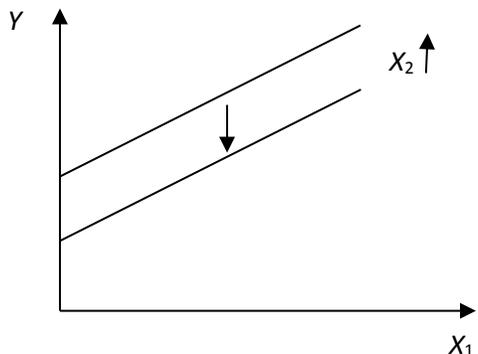
$$(A.5) \quad Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3.$$

(+) (+) (-) (+)

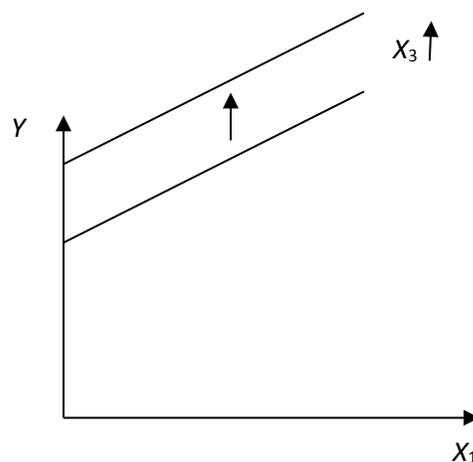
You are then asked to do the following tasks:

Task 1: Draw a graph of the relationship between Y and X_1 and show what happens to the graph when X_2 increases and then what happens if X_3 increases.

Answer: Following the steps outlined above the graphs will look as follows.



Panel A. X_1 and Y with increase in X_2



Panel B. X_1 and Y with increase in X_3

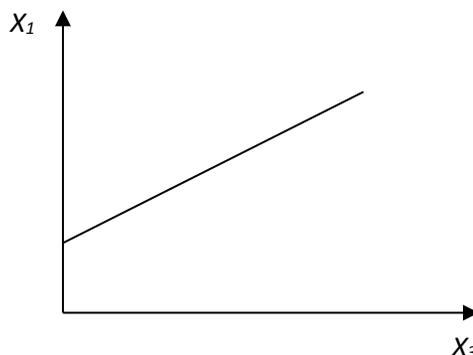
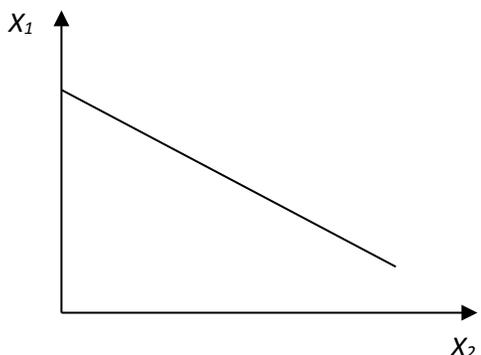
Task 2: Solve this equation for X_1 to determine the sign of the tradeoff between X_1 and X_2 and X_1 and X_3 holding Y constant (i.e., determine the isoquant equation).

Answer: Following the steps

$$X_1 = \frac{1}{b_1} Y - \frac{b_0}{b_1} - \frac{b_2}{b_1} X_2 - \frac{b_3}{b_1} X_3.$$

(+ (-) (-) (+)

Task 3: Draw the isoquant relationship between X_1 and X_2 and X_1 and X_3 holding Y and the other X constant.



Think Break!!!

1. What would the graphs look like if the task was to draw a graph of the relationship between Y and X_2 and show what happens if we decrease X_1 and then if we decrease X_3 .
2. What would the graphs look like if $b_1 < 0$?

Ok, so given a general linear **mathematical** relationship you can show it **graphically**. Can you show it **numerically**? That is, can you create a table of different values of Y with different values of the X s. **NO**. Why? Because you do not know the actual values of the coefficients; the b_0 , b_1 , etc. To actually generate numbers using the equation you need actual values of the coefficients *and* values of the X variables.

Think Break!!! Given the following equation, answer the following questions:

$$Y = 5 + X_1 - .5X_2$$

- a. What is the value of b_0 , b_1 , and b_2 in this case?
- b. How would you write this in the more general notation of equation (A.5)?

c. Fill out the following table using this equation

X_1	X_2	Y
10	10	
15	10	
20	20	
25	20	

d. Draw the graph for this data of Y and X_1 . Explain why it looks this way.

e. Draw the graph for this data of Y and X_2 . Explain why it looks this way.

f. Solve this equation for X_2 and draw its graph.

IV. Nonlinear Relationships

If you have gotten to this point and understand the linear variable relationship between graphs and functions, then the more general notation should be a snap. To this point we have been assuming the relationship between the independent variables and dependent variable is linear. Stated another way, we have been assuming as the independent variable changes the dependent variable changes at a constant rate (i.e. the slope coefficients are constant). However, many relationships are not linear but rather are nonlinear. A **nonlinear** relationship is one where the dependent variables changes at a different rate as the independent or explanatory variable changes. Here we begin as in the previous section with two dimensional relationships (two variables) before turning to multidimensional relationships.

Nonlinear Relationships with Two Variables

A growth function is a good example of a nonlinear relationship between two variables, of which the BMI growth charts are a specific example. In this case the two variables are BMI and age.

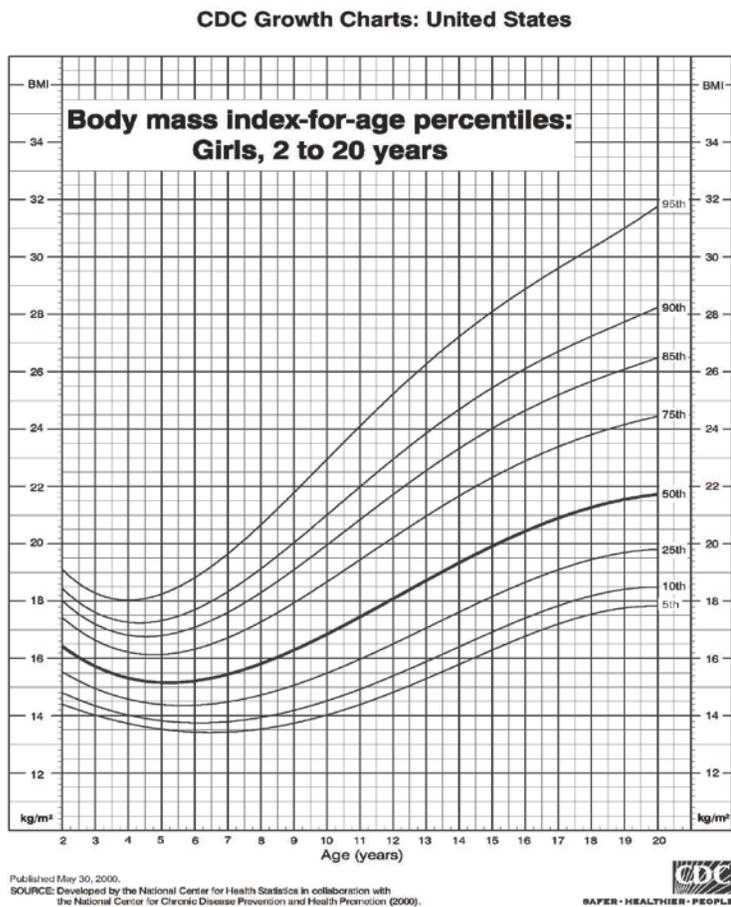
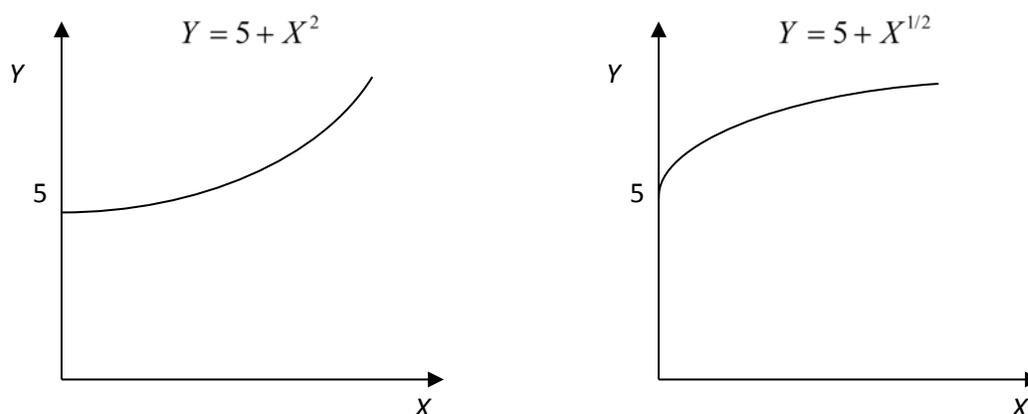


Figure A.4. Body mass index-for-age percentiles: Girls, 2 to 20 years

Consider the dark 50th percentile curve in figure A.4. Notice that the BMI actually declines until about 5.5 years of age (i.e., the relationship between age and body mass index is negative).

However, from 5.5 to about 17 years of age the BMI increases at an increasing rate and then the rate of increase slows down. Thus the rate of change (or slope) of the relationship between age and BMI is not constant and therefore by definition the relationship is not linear. This is a **graphical** presentation of a nonlinear relationship. What would a nonlinear equation look like?

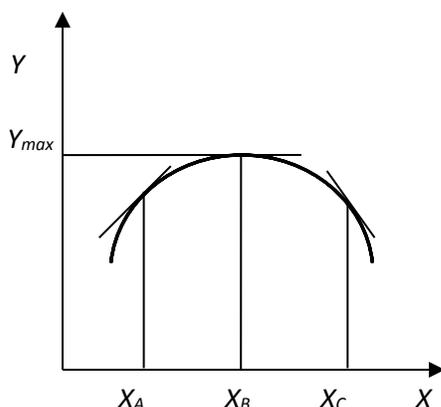
Here are two examples of nonlinear **equations** along with their **graphs**.



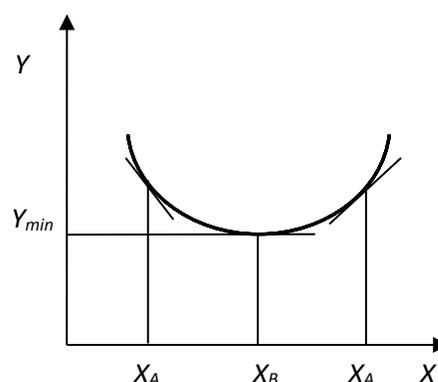
In the left panel, the value of Y is *increasing at an increasing rate* as X increases. That is, the slope increases as X increases. In the right panel, the value of Y is *increasing at a decreasing rate* as X increases. In this case, the slope decreases as X increases.

Nonlinear relationships are very important in science and especially in economics as there are many relationships we wish to consider that are nonlinear. Two of the more versatile classes of functions are concave functions and convex functions. A **concave** function is one where the slope changes at a decreasing rate. A **convex** function is one where the slope changes at an increasing rate. The panels below show a concave function and a convex function.³

³ Technically these are examples of *strictly* concave and convex functions.



Panel A. Concave Function



Panel B. Convex Function

Consider the concave function in panel A. Note up to X_B the value of Y is increasing. However, *the rate* at which Y is increasing is *decreasing*. That is, *the slope is decreasing* as X increases up to point X_B . At point X_B the value of Y is at the maximum (Y_{\max}). Note also at the maximum the slope of the function is zero! Beyond point X_B the value of Y is decreasing. The *rate* at which Y is decreasing is *decreasing*. This may seem unintuitive but you must remember that the slope is negative and it is getting more negative (e.g. going from -5 to -7). Because it is getting more negative it is decreasing. If it were getting less negative (moving toward zero) then it would be increasing (e.g., -7 to -5). But yes, the absolute value of the slope is increasing.

Now consider the convex function in panel B. Note up to X_B the value of Y is decreasing. However, the *rate* at which Y is decreasing is *increasing*. That is, *the slope is increasing* as X increases up to point X_B . Again, this may seem unintuitive but you must remember that the slope is negative but it is getting less negative (e.g., -7 to -5). Because it is getting less negative it is increasing. At point X_B the value of Y is at the minimum (Y_{\min}). Note also at the minimum the

slope of the function is zero.⁴ Beyond point X_B the value of Y is increasing. The *rate* at which Y is increasing *is increasing*. The *slope is increasing*.

The concave and convex functions are so useful because they can be combined to represent several types of relationships. The most common combined form (i.e., having concave and convex components) we will use in this class is so common that it is often referred to as the “three stage” production function, which looks a lot like the BMI growth function. This concept is first encountered in chapter 3 of the book (pp. 52-53).

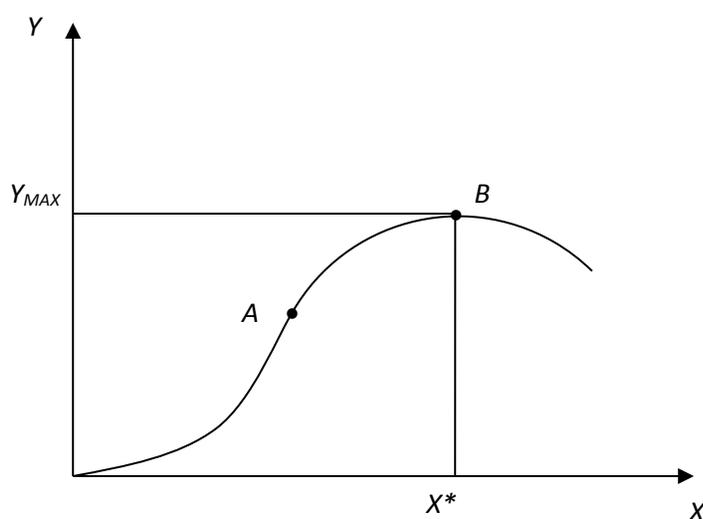


Figure A.5. Three Stage Production Function

Up to point A this function is convex. Beyond point A this function is concave. By definition we then know that up to point A , Y is increasing at an increasing rate as X increases (i.e., the slope is increasing). This is called stage one and is often referred to as the increasing returns stage.

⁴ The astute reader probably notices that the maximum value and minimum value each occur where the slope is zero. A slope equal to zero is a necessary condition for both a maximum and minimum, but it is not sufficient. Sufficient conditions relate to the change in the slope as we move away from the maximum (minimum) point and will not be considered in this course. For most of the applications in this class we will be working with strictly concave or strictly convex functions and in these cases the slope equal to zero is necessary and sufficient to denote a maximum for the strictly concave function and a minimum for the strictly convex function.

Between points A and B , Y is still increasing but at a decreasing rate as X increases (the slope is decreasing). This is called stage two and is often called decreasing returns. Point B represents where Y is at its maximum value. The value of X where Y is at its maximum is denoted as X^* .

Beyond point B is stage three.

What does this have to do with nutrition and why is it useful? This basic graph underscores the basic idea underlying the nutrient recommendations. Over a certain range (up to point A) we expect the benefits to eating more of a certain nutrient to be increasing but beyond that point we expect the benefits to not be as dramatic. At some point we reach the optimal level of the nutrient and beyond that point the more of the nutrient starts to possibly deteriorate health and could become toxic.

So what type of general notation are we going to use to denote nonlinear relationships like we did in equation (A.2)? There are an infinite number of nonlinear functions so we are not going to write out what mathematicians call the “functional form.” Instead we are just going to use general function notation. What is general function notation? **General function notation** is a simple and concise way of just saying one variable depends on another variable without indicating if the actual relationship is linear or nonlinear. For example we can say BMI increases with Age after the age of about five, as indicated by the growth charts. However, we can write this more concisely and compactly as $BMI = f(Age)$, which is literally read as ‘ BMI is a function of Age .’ Of course this does not indicate what the relationship is between BMI and Age . To remedy this we will use the same technique as before and write this as

$$BMI = f(Age).$$

(+)

NOTE: Remember the (+) sign under the variable *Age* does **not** mean age is positive or that age is increasing. It simply means there is a positive relationship between age and BMI. As age increases BMI increases. As age decreases BMI decreases.

Now the observant reader will probably be saying, ‘yeah this tells us the relationship is positive but it does not tell us if the slope is increasing or decreasing over the entire range.’ You are right, so to handle the four possible cases: (i) the relationship is always positive, but the positive slope may vary in magnitude, (ii) the relationship is always negative, but the negative slope may vary in magnitude, (iii) the relationship may be positive over the first segment and then negative over the second segment, and (iv) the relationship may be negative over the first segment and then positive over the second segment, we will use the following notation.

$$(A.5) \quad Y = f(X) \quad \text{or} \quad Y = f(X) \quad \text{or} \quad Y = f(X) \quad \text{or} \quad Y = f(X).$$

$$\quad \quad \quad (+) \quad \quad \quad (-) \quad \quad \quad (\pm) \quad \quad \quad (\mp)$$

IMPORTANT: In this class we will often write something like A.5 and then draw a graph that shows this relationship visually. Because this notation does not indicate if the relationship is linear or nonlinear often we will just draw the relationship as if it is linear.

Nonlinear Relationships With More Than Two Variables

As we have already discussed in the linear relationship case, we are often interested in the relationship between more than two variables. The concepts we used in the more than two variable linear case extend quite easily to the more than two variable nonlinear case.

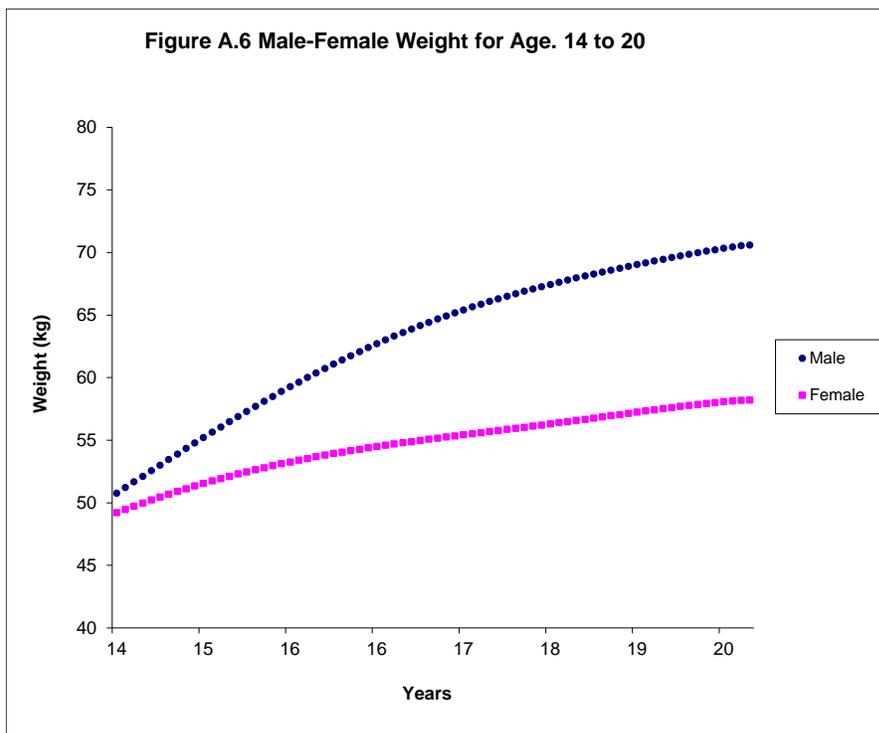
Consequently the general function notation we will use for nonlinear relationships in more than two variables is very similar to equation (A.3). So in the nonlinear case with more than two variables we will just use the following notation

$$(A.6) \quad Y = f(X_1, X_2, X_3, \dots, X_M).$$

$$\quad \quad \quad (+) \quad (-) \quad (+) \quad (\pm)$$

What does (A.6) say? It is a language stating a paragraph very concisely. It simply says that the dependent variable Y is determined by, or depends on, or is a function of M variables. As X_1 increases, Y increases. As X_2 increases, Y decreases. As X_3 increases, Y increases and etc.

Let's work in reverse and take a simple **graphical** example and then indicate how we would write it in general function notation. Figure A.6 gives the weight for age for males and females between the ages of 14 and 20.



How many variables are involved here? There are three: Weight, Age, and Gender. We note that weight and age are positively related, *ceteris paribus*. That is, holding gender constant, weight increases at a decreasing rate. What is the relationship between weight and gender? Again, holding age constant, we see that weight is higher if the gender is male. To represent these relationships **mathematically** we would write

$$(A.7) \text{ Weight} = f(\text{Age}, \text{Male}).$$

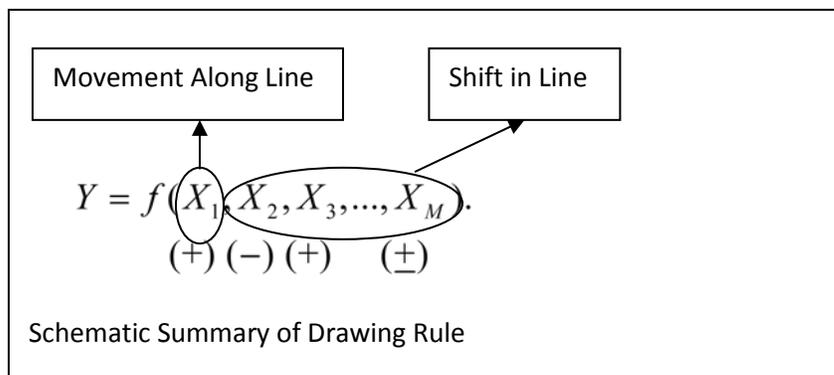
$$(+) \quad (+)$$

Equation (A.7) says that as *Age* increases (ceteris paribus) *Weight* will increase and that as *Male* increases (ceteris paribus) then *Weight* will increase. You may be wondering, “how can *Male* increase?” In this case *Male* is what we call an indicator variable. An **indicator variable** is a variable that can take on one of two values (usually 0 and 1). If the indicator variable is ‘on’ (e.g., $\text{Male} = 1$) then the relationship applies under that case. If the indicator variable is ‘off’ (e.g., $\text{Male} = 0$) then the relationship applies under the alternative case (e.g., Female). So if you were given equation (A.7) and asked to draw it, you would draw a graph like figure A.6. Note then the variable *Male* is a shift variable when drawing *Weight* versus *Age*.

So what are the general rules for **drawing a graph for a multidimensional general function** such as (A.6)? The procedure is very similar to that of the linear case:

1. Choose the *X* variable you want to show related to *Y*.
2. Label the horizontal axis *X* and the vertical axis *Y*.
3. Look at the sign under the variable with the *X* variable you chose. If it is positive then draw a line (or curve) from southwest to northeast. If it is negative draw a line (or curve) from northwest to southeast. This graph shows the relationship between the *X* you chose and *Y*.
4. To show the effect on this graph of changing some other *X* variable, look at the sign under this other *X* variable. If it is positive, then increasing this variable will shift the curve from step 3 out. If it is negative, then increasing this variable will shift the curve from step 3 in.

Schematically the process is basically the same as before.



NOTE AGAIN: Just by relabeling the variables, the first explanatory variable (X_1) can be any variable and the actual signs under the variables will depend on the function being discussed.

Nonlinear Isoquants

As we have already seen, isoquants are especially useful devices for showing tradeoffs between X variables holding the dependent variable Y constant. A good understanding of isoquants also allows us to draw three dimensional diagrams in two dimensions.

So where does the isoquant come from graphically in the nonlinear more than two variable case? Let's consider the figure A.7 below. This figure shows the **side** view of a mountain on top and then a **contour** view of the mountain on the bottom. There are three variables in this figure: the height or elevation of the mountain (let's label this Y), the North-South direction (let's label this X_1) and the East-West direction (let's label this X_2). So there are three dimensions represented by the variables Y, X_1, X_2 . As we know, and have mentioned, a piece of paper is a two dimensional surface so we effectively want to draw the relationship between the three variables in two dimensions and that is what the contour map does. Pretend you are standing at sea level (see stick figure). You want to walk to the house on the far left side of the diagram. How could you get there? Well if you walk due West (directly left) you will start climbing the mountain. As you walk you will go from sea level to 25 feet above sea level, 50 feet above sea

level, 75 feet above sea level, 100 feet above sea level, and to the top of the highest peak and then you will walk back down, once again passing 100 feet, then 75 feet, down to about 60 feet before ascending the second peak and then finally walking down the second peak until you get to the house, which is at sea level.

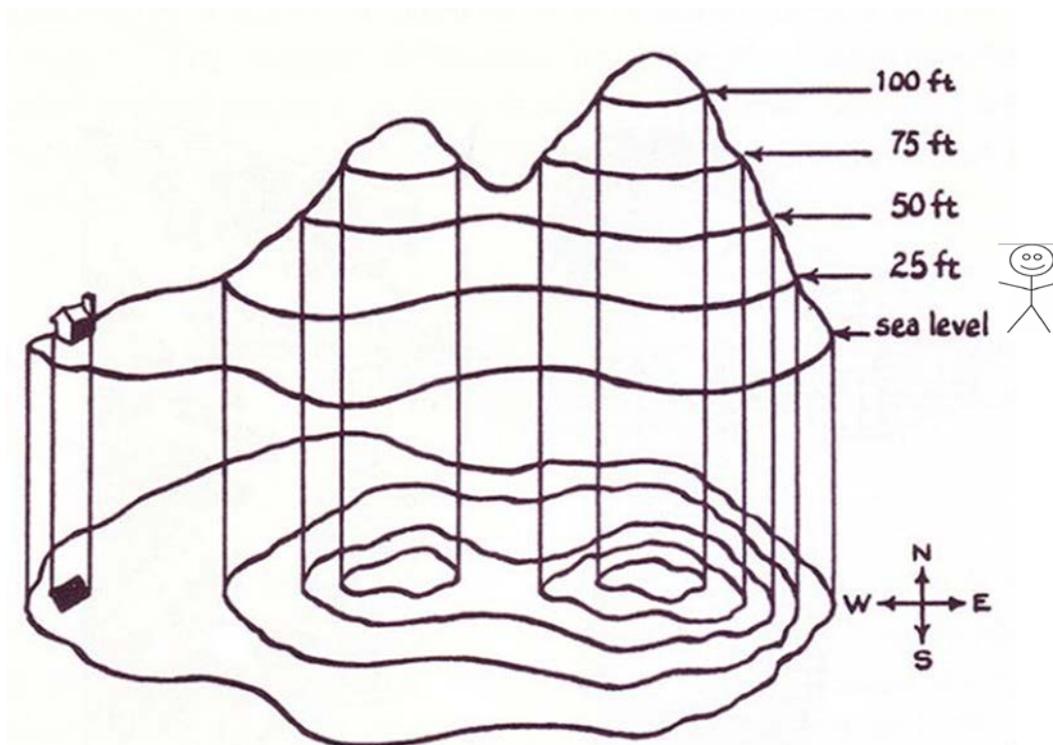


Figure A.7. Contour Map of a Mountain

So that is one way to get there. However, there is an easier way to get to the house. If you starting walking South West at sea level and just stay at sea level (walking Southwest, then West) you would also get to the house. Simply stated, you walked around the mountain and ***you stayed at the same elevation!*** Now note the contour map on the bottom provides the exact same information about the elevation of the mountain, but without showing the mountain. That is, the most outside ring represents sea level, the next inner ring represents all elevations of 25 feet, the next inner ring represents all elevations of 50 feet, etc.

Let's now apply this contour logic to the health production function from chapter 3 (p. 52-54) but which now depends on two variables, like protein (X_1) and carbohydrate (X_2) intake, where we will assume that as these increase health (Y) increases, so we will write the relationship in general nonlinear notation as

$$(A.8) \quad Y = f(X_1, X_2).$$

(+)(+)

Graphically, then we would have the health “mountain” as

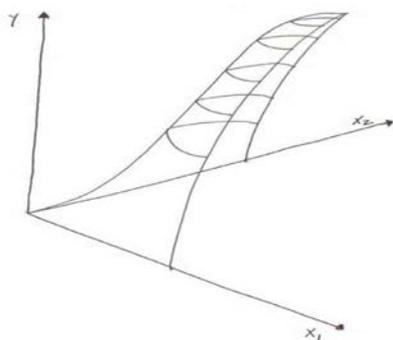


Figure A.8. Neoclassical three stage Production Function in 2 Inputs

Think of this production function like a mountain and the same ideas from figure A.7 apply. The X_1 axis runs west to east. The X_2 axis runs south to north. The Y axis runs into the sky and measures altitude. Ok, so if you start at the origin (i.e., where all intersect) and start walking northeast then you are going to start walking “up the mountain” along the ridgeline. Every step

you take going up this mountain you are taking a step north east and increasing your altitude. Note this means three things are happening or the values of three variables are changing with each step: the value of X_1 is increasing, the value of X_2 is increasing, and the value of Y is increasing.

Now suppose you get tired of going up and just want to walk around the mountain at the same altitude. In this case you would start just start walking to either the east or west. Note as you continue to walk though if you want to stay at the same altitude then eventually you will go from just walking east (or west) and must start walking north again. In mathematical terms you are walking along an isoquant (also known as a level curve or contour line). The rings on the graph represent the isoquants (i.e., where the altitude or Y is constant). Each isoquant is therefore associated with a specific altitude or value of Y .

Suppose now you come walking around the mountain and there is a guy giving helicopter rides. He says, “Do you want to see the mountain from the air?” You say “sure”, so you hop in and the pilot takes you directly over the mountain. What will it look like? It will look like figure A.9 below.

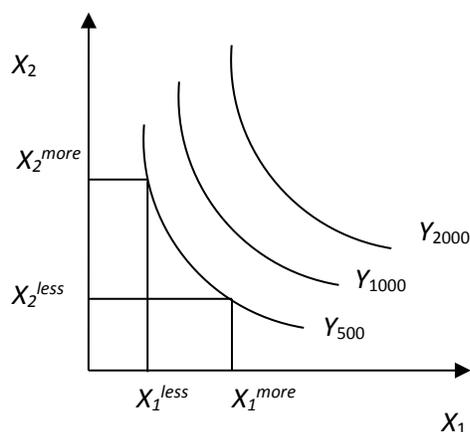


Figure A.9. Convex Isoquants

The curve labeled Y_{500} refers to all points on the mountain corresponding to 500 feet. This is the 500 foot isoquant. Note there are numerous combinations of X_1 and X_2 that yield the same altitude: (X_2^{more}, X_1^{less}) and also (X_2^{less}, X_1^{more}) . Notice then if we want to stay at the same altitude as we decrease X_2 we must increase X_1 to offset the decrease in X_2 . We therefore have the general principle:

Moving along an isoquant shows the tradeoff between two inputs holding the output constant.

This simple concept is one of the most useful and powerful in economics. A vernacular way to say it is “there is more than one way to skin a cat.”

Here are a couple of useful instructional videos on YouTube explaining where contour maps come from if you are still confused.

<https://www.youtube.com/watch?v=L1AWNR-Y0pQ>

<https://www.youtube.com/watch?v=9kNkpQTWpC0>

V. Optimization

One of the most important concepts in all of mathematics and science is optimization.

Optimization is the process of determining the optimal value in satisfying some objective.

Simply stated optimization means the “best” choice among all alternative choices. Life is about optimization. If you recall the definition of economics you now realize that optimization and economics goes hand in hand and it is for this reason that many say everything in life is about economics (i.e., choices). Choosing the “best” school, the “best” degree program, the “best” job, the “best” boyfriend, or less consequential, the “best” car, the “best” phone, the “best” wine, or the “best dessert”. Why is “best” in parenthesis? Because what is “best” for you may not be what is best for someone else because you have different values or different objectives. In addition, you may have to choose the “best” subject to conditions that others may not face. For

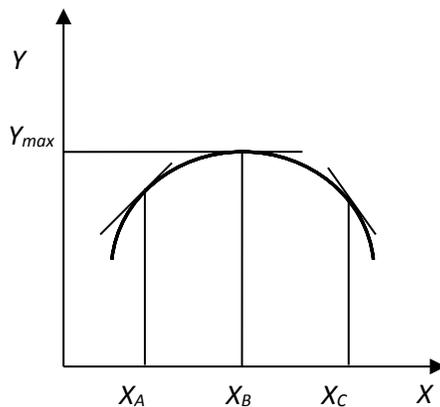
example, what is the “best” car you would choose? If you live in the city then the “best” may be a BMW 320i but if you live in the country “best” may be a Dodge Ram truck. In addition, even if you live in the city, if you only make \$20,000 a year then your “best” car may be a 2002 Mazda 626 because that is all you can afford. In this case your “best” choice is constrained by your income. This leads to the need to distinguish between unconstrained optimization and constrained optimization. **Unconstrained optimization** refers to the situation where there is no impediment or barrier to finding the optimal value. **Constrained optimization** refers to the situation where there is an impediment or barrier to reaching the unconstrained optimal value.

Note also that the term optimization can apply to maximization or minimization and each of these is just a special case of optimization. **Maximization** is the process of determining the largest value in satisfying some objective. **Minimization** is the process of determining the smallest value in satisfying some objective. An individual may want to maximize their income, but they probably want to minimize how much they spend for a car. In addition, in some cases we may be only considering the value of one variable to choose to optimize something. For example, if you are trying to maximize your income then you would choose the job with the highest salary. However, in other situations you may be trying to consider two or more variables to choose to optimize something. For example, every time you eat a meal you are deciding the unique combinations of solid food and liquid to consume until you are satisfied. In this section we discuss unconstrained optimization and constrained optimization for one and two variables focusing on maximization and minimization.

Unconstrained Optimization with One Choice Variable

Here we cover unconstrained optimization with one choice variable for maximization and minimization problems.

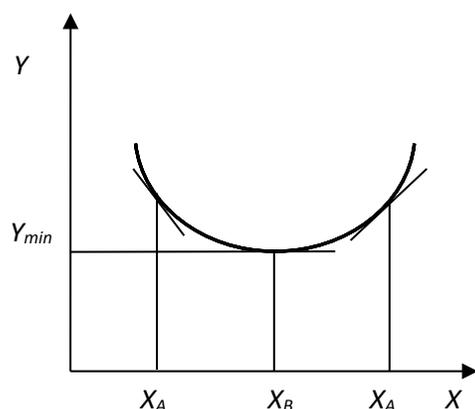
Unconstrained maximization with one choice variable is the case where there is a single variable whose value can be chosen to maximize some function. We have already seen a graph for maximization and it is repeated below in Figure A.10. Note up to X_B the value of Y is increasing. This means the slope is positive and though the rate at which Y is increasing is decreasing the value of Y is still increasing for every additional unit of X added. At point X_B the value of Y is at the maximum (Y_{\max}). Using function notation we would write this maximum point as $Y_{\max} = f(X_B)$. Note also at the maximum the slope of the function is zero! Beyond point X_B the value of Y is decreasing. Thus if you choose X beyond X_B the value of Y starts to decrease. A good example of this would be some type of nutrient. Up to some point the health benefits are increasing but beyond that point the health benefits may begin to decline as you have reached a toxic level of the nutrient.



**Figure A.10 Maximization
One Choice Variable**

Unconstrained minimization with one choice variable is the case where there is a single variable whose value can be chosen to minimize some function. A graph of a function to minimize has already been shown and is repeated below in Figure A.11. Note up to X_B the value of Y is decreasing. This means the slope is negative and though the rate at which Y is decreasing is decreasing the value of Y is still decreasing for every additional unit of X added. At point X_B

the value of Y is at the minimum (Y_{\min}). Using function notation we would write this minimum point as $Y_{\min} = f(X_B)$. Note also at the minimum the slope of the function is zero! Beyond point X_B the value of Y is increasing. Thus if you choose X beyond X_B the value of Y starts to increase. A good example of this type of relationship may be average food costs per person in a household related to number of individuals in the household. The *total cost* of groceries will increase as the number of individuals in the household increases. However, the *cost per person* (average cost) may actually decrease up to some point (say 4 people) before it starts to increase again.

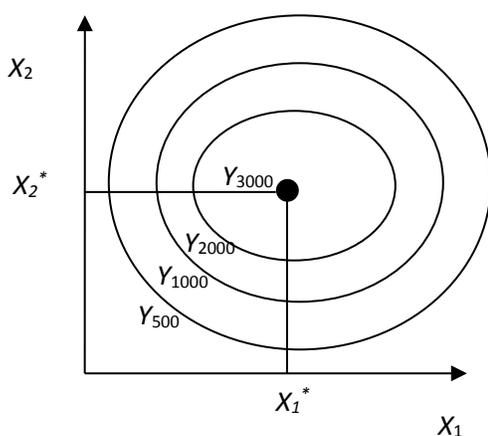


**Figure A.11. Minimization
One Choice Variable**

Unconstrained Optimization with Two Choice Variables

In the above cases, the value of Y only depended on one variable, X . However, in many cases the value of some variable of interest will often depend on more than one variable. For example, we know that a gram of carbohydrates is worth 4 calories and a gram of fat is worth 9 calories. Thus someone's weight could be written as weight (Y) is a function (or depends on) the amount of carbohydrates (X_1) and the amount of fat (X_2), *along with many other things we are ignoring at this point*. In function notation we write this "production" function as $Y = f(X_1, X_2)$.

Unconstrained maximization with two choice variables is the case where there are two variables whose values can be chosen to maximize some function. Think again of climbing the mountain we drew in figure A.9 but now let's draw the complete mountain top as shown in figure A.12.

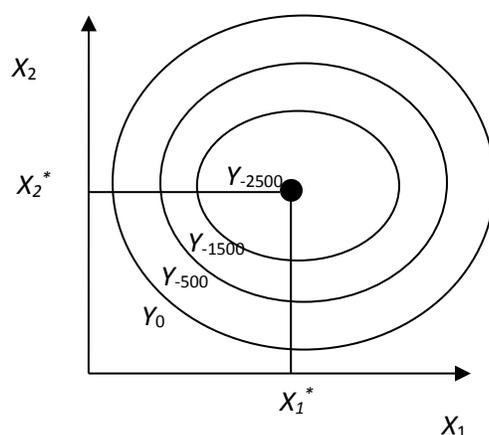


**Figure A.12. Maximization
Two Choice Variables**

This is the view from the helicopter of the top of the mountain. As before, the notation Y_{500} denotes the isoquant or level curve where the altitude is 500 feet and the entire outer circle is 500 feet. The next inner circle denotes the isoquant of 1000 feet (Y_{1000}), followed by the 2000 foot isoquant (Y_{2000}) and the top of the mountain, which is 3000 feet (Y_{3000}). Note to get to the top of this mountain you must go to the coordinates (X_1^*, X_2^*) . What is true at this point? Well because you are at the top (you can't go any higher), then the slope at the top is zero in *all* directions. In addition, or alternatively stated, any other coordinate combinations will put you at a lower altitude!! In function notation we will write this maximum point as $Y_{\max} = f(X_1^*, X_2^*)$.

Unconstrained minimization with two choice variables is the case where there are two variables whose values can be chosen to minimize some function. Think now of climbing down

into a gorge or big hole. We can use a graph similar to figure A.12 to show the minimization as shown in figure A.13.



**Figure A.13. Minimization
Two Choice Variables**

This is the view from the top of the hole. As before, the notation Y_0 denotes the isoquant or level curve where the altitude is zero feet or ground level and the entire outer circle is 0 feet. The next inner circle denotes the isoquant of -500 feet (Y_{-500}), followed by the -1500 foot isoquant (Y_{-1500}) and the bottom of the hole, which is -2500 feet (Y_{-2500}). Note to get to the bottom of this hole or gorge you must go to the co-ordinates (X_1^*, X_2^*) . What is true at this point? Well because you are at the bottom (you can't go any lower), then the slope at the bottom is zero in *all* directions. In addition, or alternatively stated, any other coordinate combinations will put you at a higher altitude!! In function notation we will write this maximum point as $Y_{\min} = f(X_1^*, X_2^*)$.

Constrained Optimization with One Choice Variable

In most cases optimization cannot occur without some barriers or impediments. As mentioned earlier, you may *want* a 2009 BMI 320i for your car and if money were no object then that would be what you would buy. However, money is an object and what you are *able* to buy is not the same as what you may *want* to buy. Your income constrains what you can afford to

buy. So we immediately see what may be optimal in an unconstrained setting (a 2009 BMI 320i) is infeasible in a constrained setting.

Constrained optimization is very easy to see graphically. Suppose we are considering “producing” a good score on an exam (Y) and the input in this production function is time studying (X). Now suppose you go to class six hours a day (including commute time and other incidental time) and you work at the local coffee shop six hours a day. That leaves you 12 hours for sleeping, eating, personal hygiene, studying, and leisure. You have determined that you can spend at most six hours studying for the exam. So the range of X is now restricted to be 0 – 6 hours, or in mathematical terms $X \leq 6$ hrs. How can we show this graphically? Well using the three stage production function graph from earlier we can show this as in figure A.14. Studying the full 6 hours allows you to reach the constrained score (S_C). However, if this constraint were not in place or not binding, then you could have studied T_U and achieved the optimal unconstrained score of S_U .

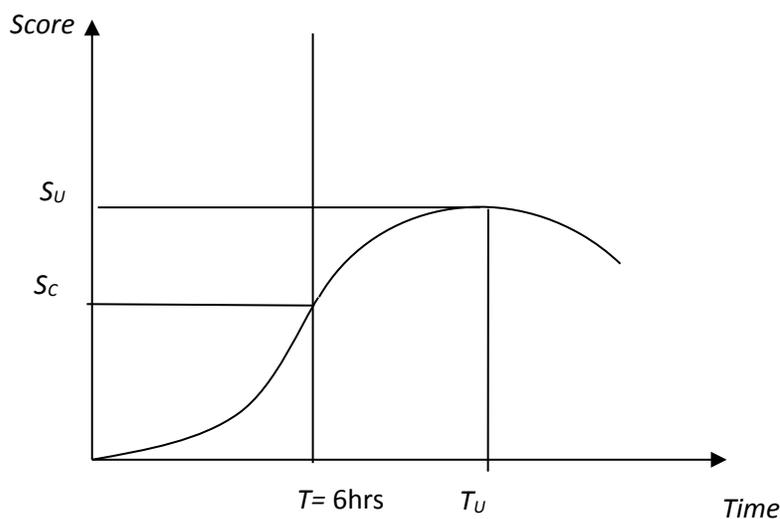


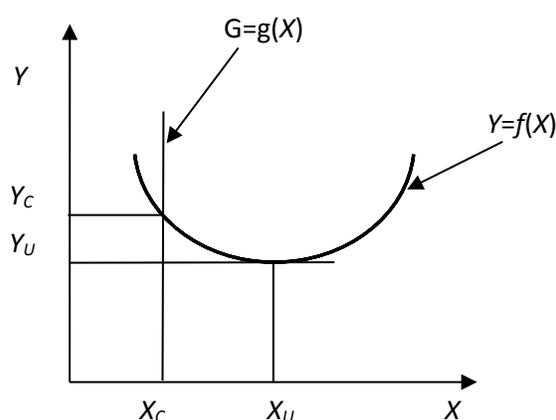
Figure A.14. Three Stage Grade Production Function with Constrained Time Input

THINK BREAK!!!!

- What happens in figure A.14 if the time constraint is relaxed to say $6 \leq X \leq 10$?
- What happens in figure A.14 if the time constraint is relaxed to say $X > T_U$?

The constrained minimization problem is similar. Just change the word ‘maximize’ to ‘minimize.’ **Constrained minimization with one choice variable** is the case where there is a single variable whose value can be chosen to minimize some function subject to a constraint.

A graphical representation of a constrained minimization problem could look like figure A.15.



**Figure A.15. Constrained Minimization
One Choice Variable**

In figure A.15 the constrained minimum point is (X_C, Y_C) whereas the unconstrained minimization point is (X_U, Y_U) .

Constrained Optimization with Two Choice Variables

Just as a constraint can prevent you from reaching the optimal value when there is a single variable to choose, the same concept applies in the more general case: a constraint in a two variable choice context prevents you from reaching the unconstrained optimum. Let’s again consider this graphically first. Suppose you are once again walking up the mountain as before in

figure A.12. However, now there is a man named G.O. Away who owns the top part of the mountain and he has built a wall than runs from the Northwest to the Southeast. From the **helicopter view**, the mountain with the wall looks like figure A.16

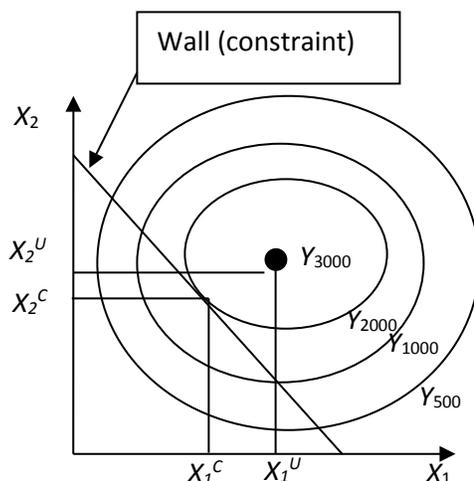


Figure A.16. Constrained Maximization Two Choice Variables

Now suppose you start walking up the mountain with the intention of going to the highest point. You start walking from the origin to the Northeast. You pass 500 feet and then you pass 1000 feet but at 2000 feet you run into the wall at the co-ordinates (X_1^C, X_2^C) . You decide to attempt to walk around the wall and get higher and decide to walk Southeast along the wall. However, as you walk Southeast you realize your altitude is decreasing. You keep walking until you reach the point where X_1^U intersects the wall and at this point the altitude is 1000 feet. You then decide to go back and walk Northwest along the wall. As you go back toward your original starting point (X_1^C, X_2^C) the altitude is increasing. Encouraged you continue to walk Northwest along the wall. However, once you pass your original starting point (X_1^C, X_2^C) the altitude starts to decrease again. You conclude (correctly) that the highest point you can reach on the mountain given the wall is at point (X_1^C, X_2^C) . What is unique about this point? At this point, the wall is tangent to the isoquant for 2000 feet. Recall, **tangent** means the slope of one line is equal to the

slope of another line. In this case, the slope of the wall is equal to the slope of the isoquant at 2000 feet.

THINK BREAK!!!!

- Using figure A.16 as a guide, can you use figure A.13 to draw the constrained minimization graph and explain?

VI. Conclusions

The purpose of this mathematical appendix is to help you “brush off the math rust” and provide you a general overview of the main mathematical concepts that are used in the book. As indicated in the book and here, mathematics is best considered a language for communicating concepts. Like all languages it is better at communicating some ideas than others. The appeal of the mathematical language is its precision in discussing, describing, and explaining relationships between variables, their logical connections, and their logical implications. The precision is extremely helpful in avoiding logical errors that easily occur when using a less precise language (e.g. words) that is not well suited for understanding and deriving logical implications from the interactions of variables. The reader who is comfortable with the ‘language’ of this appendix will find the material in the book much easier to understand, will travel much further in their journey through the land of economics, and will gain a much deeper appreciation for its advantages and disadvantages in addressing issues related to food and nutrition.